

Hermitian Forms Meet Several Complex Variables

Minicourse on CR Geometry Using Hermitian Forms

by Jiří Lebl

May 19, 2026

Typeset in L^AT_EX.

Copyright ©2011,2026 Jiří Lebl



This work is licensed under the Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/3.0/us/> or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA.

You can use, print, duplicate, share these notes as much as you want. You can base your own notes on these and reuse parts if you keep the license the same. If you plan to use these commercially (sell them for more than just duplicating cost), then you need to contact me and we will work something out. If you are printing a course pack for your students, then it is fine if the duplication service is charging a fee for printing and selling the printed copy. I consider that duplicating cost.

During the writing of these notes, the author was in part supported by NSF grant DMS-0900885.

See <https://www.jirka.org/scv-mini/> or <https://jirilebl.github.io/scv-mini/> for more information (including contact information).

Contents

Introduction	4
0.1 Notes about these notes	4
1 Complex Analytic Varieties	5
1.1 Holomorphic functions	5
1.2 Weierstrass preparation and division theorems	9
1.3 The ring of germs	13
1.4 Varieties	14
2 CR Geometry	20
2.1 Proper maps and the inequivalence of ball and polydisc	20
2.2 Polarization	22
2.3 Hermitian forms for polynomials	23
2.4 Complex varieties in real-algebraic hypersurfaces	26
2.5 Hermitian forms for real-analytic functions	30
2.6 Complex varieties in real-analytic hypersurfaces	33
3 Proper Maps of Balls	37
3.1 Proper maps of balls	37
3.2 Tensors and polynomial proper maps	40
3.3 Monomial proper maps	42
3.4 Projective setting and hyperquadrics	44
3.5 Hermitian forms	46
3.6 Degree 2 maps	49
3.7 Monomial degree estimates	53
Further Reading	58
Bibliography	58
Index	61

Introduction

0.1 Notes about these notes

These were the lecture notes for a graduate Math 595 half-semester mini-course at University of Illinois at Urbana-Champaign on Several Complex Variables given in Spring 2010. Prerequisites are one variable complex analysis, linear algebra, real analysis and basic functional analysis. The notes are not meant as a polished finished product. I hope they are usable as a quick and dirty introduction to the subject. Due to time constraints much is missing. For example, neither the Levi form nor pseudoconvexity is mentioned. The focus is on geometry using Hermitian forms rather than analysis.

The notes are also not completely self contained. There are many proofs in the beginning sections about complex geometry that are left out due to lack of time in the course. Some material from the course did not make it into these notes and vice versa.

There are most likely many typos and minor errors throughout (hopefully no major ones), so be wary. Do email me any errors you find so that they get fixed. See <https://www.jirka.org/scv-mini/> or <https://jirilebl.github.io/scv-mini/> for more info.

The structure of the course is the following.

1. *Complex Analytic Varieties*. This chapter covers the very basics of complex geometry and holomorphic functions of several variables.
2. *CR Geometry*. We solve some basic problems in CR geometry using Hermitian forms. The chapter begins with the inequivalence of the ball and polydisc in \mathbb{C}^2 for motivation.
3. *Proper Maps of Balls*. A very natural place where Hermitian forms arise are proper maps of balls and hyperquadrics. We cover this topic in this chapter.

I would like to acknowledge my SCV minicourse class at UIUC for any comments during the class, and especially Dusty Grundmeier. I would also like to acknowledge Montgomery Taylor and Jianou Zhang for reading through and finding errors.

Chapter 1

Complex Analytic Varieties

1.1 Holomorphic functions

Let \mathbb{C}^n denote the complex Euclidean space. We denote by $z = (z_1, \dots, z_n)$ the coordinates of \mathbb{C}^n . Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ denote the coordinates in \mathbb{R}^n . We can identify \mathbb{C}^n with $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by letting $z = x + iy$. Just as in one complex variable we write $\bar{z} = x - iy$. We call z the *holomorphic coordinates* and \bar{z} the *antiholomorphic coordinates*.

Definition 1.1.1. For $\rho = (\rho_1, \dots, \rho_n)$ where $\rho_j > 0$ and $a \in \mathbb{C}^n$ define a *polydisc*

$$\Delta_\rho(a) \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n : |z_j - a_j| < \rho_j\}. \quad (1.1)$$

We call a the *center* and ρ the *polyradius* or simply the *radius* of the polydisc $\Delta_\rho(a)$.

At this point it will be useful to recall the *Euclidean inner product*

$$\langle z, w \rangle = z \cdot \bar{w} = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n. \quad (1.2)$$

Using the inner product we obtain the standard *Euclidean norm*

$$\|z\| \stackrel{\text{def}}{=} \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \dots + |z_n|^2}. \quad (1.3)$$

As in complex analysis in one variable we define the formal operators

$$\frac{\partial}{\partial z_j} \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad (1.4)$$

$$\frac{\partial}{\partial \bar{z}_j} \stackrel{\text{def}}{=} \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right). \quad (1.5)$$

Definition 1.1.2. Let $U \subset \mathbb{C}^n$ be an open set, and let $f: U \rightarrow \mathbb{C}$ be a continuously differentiable function. We say that f is *holomorphic* if it satisfies the Cauchy-Riemann equations

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \quad \text{for } j = 1, \dots, n. \quad (1.6)$$

In other words, f is holomorphic if it is holomorphic in each variable separately as a function of one variable. Suppose that f is holomorphic in some polydisc $\Delta = \Delta_\rho(a) = \Delta_1 \times \dots \times \Delta_n$ centered at a , where $\Delta_j \subset \mathbb{C}$ is a disc. Suppose that f is continuous on the closure $\bar{\Delta}$. We can apply the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta_1} \frac{f(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} d\zeta_1. \quad (1.7)$$

We apply the formula $n - 1$ more times and write the integral as a single integral. The operation requires the Fubini theorem, which applies as f is continuous on the closed polydisc. We obtain the Cauchy integral formula in several variables.

Theorem 1.1.3 (Cauchy integral formula). *Let Δ be a polydisc centered at $a \in \mathbb{C}^n$. Suppose that $f: \bar{\Delta} \rightarrow \mathbb{C}$ is a continuous function holomorphic in Δ . Write $\Gamma = \partial \Delta_1 \times \dots \times \partial \Delta_n$. Then for $z \in \Delta$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\zeta_1, \zeta_2, \dots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \dots (\zeta_n - z_n)} d\zeta_1 \wedge d\zeta_2 \wedge \dots \wedge d\zeta_n. \quad (1.8)$$

We will use the so-called *multi-index notation*. Let $\alpha \in \mathbb{Z}^n$ be a vector of integers. We write

$$z^\alpha \stackrel{\text{def}}{=} z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} \quad (1.9)$$

$$\frac{1}{z} \stackrel{\text{def}}{=} \frac{1}{z_1 z_2 \dots z_n} \quad (1.10)$$

$$dz \stackrel{\text{def}}{=} dz_1 \wedge dz_2 \wedge \dots \wedge dz_n \quad (1.11)$$

$$|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \alpha_2 + \dots + \alpha_n \quad (1.12)$$

$$\alpha! \stackrel{\text{def}}{=} \alpha_1! \alpha_2! \dots \alpha_n! \quad (1.13)$$

In this notation, the Cauchy formula becomes

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)} d\zeta. \quad (1.14)$$

If we differentiate both sides (assuming $\alpha \in \mathbb{N}_0^n$, i.e. $\alpha_j \geq 0$) we obtain

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{\alpha! f(\zeta)}{(\zeta - z)^{\alpha+1}} d\zeta, \quad (1.15)$$

where $\alpha + 1 = (\alpha_1 + 1, \dots, \alpha_n + 1)$. Just as in one-variable theory, the formulas imply that f is infinitely differentiable and furthermore if f_j converges to f uniformly on compact subsets, then $\frac{\partial^{|\alpha|} f_j}{\partial z^\alpha} \rightarrow \frac{\partial^{|\alpha|} f}{\partial z^\alpha}$ uniformly on compact subsets.

Let $\|f\|_K$ denote the supremum norm of f over the set K . Taking absolute values and suprema we obtain in the same way as in one-variable theory:

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) \right| \leq \frac{\alpha!}{\rho^{|\alpha|}} \|f\|_\Gamma. \quad (1.16)$$

These are the so-called *Cauchy estimates*.

As in one variable theory, holomorphic functions have a power series expansion.

Theorem 1.1.4. *Let $\Delta = \Delta_\rho(a)$. Suppose that $f: \bar{\Delta} \rightarrow \mathbb{C}$ is continuous and holomorphic on Δ . Then on Δ , f is equal to a series converging uniformly on compact subsets of Δ :*

$$f(z) = \sum_{\alpha} c_{\alpha} (z-a)^{\alpha}. \quad (1.17)$$

Conversely if f is defined by (1.17) converging uniformly on compact subsets of Δ , then f is holomorphic on Δ .

Proof. First assume that f is holomorphic. We write the kernel of the Cauchy formula as

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} \frac{1}{\left(1 - \frac{z-a}{\zeta-a}\right)}. \quad (1.18)$$

We expand in a multiple geometric series. We can then integrate termwise to obtain (1.17) where

$$c_{\alpha} = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}} d\zeta. \quad (1.19)$$

The details are left to the reader.

The converse follows by applying the Cauchy-Riemann equations to the series termwise. \square

Theorem 1.1.5 (Identity theorem). *Let $U \subset \mathbb{C}^n$ be a connected open set and let $f: U \rightarrow \mathbb{C}$ be holomorphic. Suppose that $f|_N \equiv 0$ for an open subset $N \subset U$. Then $f \equiv 0$.*

Proof. Let Z be set where all derivatives of f are zero; then $N \subset Z$. The set Z is closed in U as all derivatives are continuous. Take an arbitrary $a \in Z$. We find $\Delta_\rho(a) \subset U$. Expand f in a power series around a . As the coefficients are given by derivatives of f , we see that the power series is identically zero and hence f is identically zero in $\Delta_\rho(a)$. Therefore Z is open in U and $Z = U$. \square

Of course this also means that if we have two functions f and g such that $f = g$ on a small open set, then $f \equiv g$. We also have the maximum principle from one variable theory. We prove this using a useful technique that can be used for generalizing all sorts of basic results to several variables.

Theorem 1.1.6 (Maximum principle). *Let $U \subset \mathbb{C}^n$ be a connected open set. Let $f: U \rightarrow \mathbb{C}$ be holomorphic and suppose that $|f(z)|$ attains a maximum at some $a \in U$. Then $f \equiv f(a)$.*

Proof. The set E of points $z \in U$ where $f(z) = f(a)$ is closed. Let $b \in E$. We look at the complex line $L = \{b + \lambda v : \lambda \in \mathbb{C}\}$ for a $v \in \mathbb{C}^n \setminus \{0\}$. The one-variable function $\lambda \mapsto f(b + \lambda v)$ is holomorphic. If we look at the connected component V of $L \cap U$ that contains b , we can apply the one variable Maximum principle and obtain that $f|_V \equiv f(b) = f(a)$. As this is true for all complex lines through b , we can see that $f(z) = f(a)$ for a small open neighborhood around b . Thus E is open and therefore $E = U$. \square

Definition 1.1.7. Let $U \subset \mathbb{C}^n$ be an open set. Define $\mathcal{O}(U)$ to be the ring of holomorphic functions.

Exercise 1.1.1: *Show that $\mathcal{O}(U)$ is an integral domain (has no zero divisors) if and only if U is connected.*

For a holomorphic function f we can reorder the series and write

$$f(z) = \sum_{k=0}^{\infty} f_k(z-a), \quad (1.20)$$

where f_k is homogeneous polynomial of degree k , that is, the monomials of f_k are all of degree k .

Definition 1.1.8. For $a \in \mathbb{C}^n$ such that f is a holomorphic function defined near a , define

$$\text{ord}_a f \stackrel{\text{def}}{=} \min\{k \in \mathbb{N}_0 : f_k \neq 0\}, \quad (1.21)$$

with the obvious definition that if $f \equiv 0$, then $\text{ord}_a f = \infty$. The number $\text{ord}_a f$ is called the *order of vanishing* of f at a .

A useful concept for study of local geometry will be that of germs of functions.

Definition 1.1.9. Let p be a point in a topological space X . Let Y be a set and $U, V \subset X$ be open neighborhoods of p . We say that two functions $f: U \rightarrow Y$ and $g: V \rightarrow Y$ are equivalent if there exists a neighborhood Ω of p such that $f|_{\Omega} = g|_{\Omega}$.

An equivalence class of functions defined in a neighborhood of p is called a *germ of a function*. Usually it is denoted by (f, p) , but we may simply say f when the context is clear.

Germs are particularly useful for analytic functions because of the identity theorem. The set of germs of complex valued functions always forms a ring. When the functions are holomorphic, the ring has many nice properties.

Definition 1.1.10. Let $p \in \mathbb{C}^n$. Define ${}_n\mathcal{O}_p = \mathcal{O}_p$ to be the ring of germs of holomorphic functions near p .

Exercise 1.1.2: *Show that \mathcal{O}_p is an integral domain (has no zero divisors).*

Exercise 1.1.3: Show that \mathcal{O}_p is isomorphic to the ring of convergent power series.

Exercise 1.1.4: Show that given a germ (f, p) of a holomorphic function, we can always pick an open neighborhood U of p and a representative $f: U \rightarrow \mathbb{C}$ such that any other representative g of (f, p) will extend to be a holomorphic function on an open set containing U such that $g|_U \equiv f$.

Similarly we also have germs of sets.

Definition 1.1.11. Let p be a point in a topological space X . We say that sets $A, B \subset X$ are equivalent if there exists a neighborhood N of p such that $A \cap N = B \cap N$.

An equivalence class of sets is called a *germ of a set at p* . Usually it is denoted by (A, p) , but we may simply say A when the context is clear.

The notation $(X, p) \subset (Y, p)$ is now defined in an obvious manner. So can the operation of union and intersection of germs be defined.

Exercise 1.1.5: Write down a rigorous definition of subset, union, and intersection of germs. Then check that what you did is well defined.

It will be on occasion useful to use the implicit and inverse function theorems in the holomorphic category. Their statements are essentially the same as in the real category except that everything is holomorphic. For this reason we assign the following exercise.

Exercise 1.1.6: Give a holomorphic version of the implicit and inverse function theorems. You can use the statements from the real category, you simply need to show that if the function we are starting with is holomorphic, then the function we end up with is also holomorphic.

Remark 1.1.12. It is sometimes useful to work with \bar{z} instead of z . A function f such that $\frac{\partial f}{\partial \bar{z}} = 0$ is said to be *antiholomorphic* and can be written locally as a power series in \bar{z} . Taking the concept further, we can have a function be holomorphic in certain variables and antiholomorphic in others. Such a concept will come up when we talk of polarization in section 2.2.

1.2 Weierstrass preparation and division theorems

We wish to study the zero sets of holomorphic functions. A basic tool in the subject are the so-called Weierstrass preparation and division theorems.

1.2.1 The preparation theorem

Definition 1.2.1. Let $U \subset \mathbb{C}^{n-1}$ be open. Let $z' \in \mathbb{C}^{n-1}$ denote the coordinates. Suppose that $p \in \mathcal{O}(U)[z_n]$ is monic of degree k , that is,

$$p(z', z_n) = z_n^k + \sum_{j=0}^{k-1} c_j(z') z_n^j. \quad (1.22)$$

Further suppose that $c_j(0) = 0$ for all j . Then p is called a *Weierstrass polynomial* of degree k .

The purpose of this section is to show that every holomorphic function in $\mathcal{O}(U)$ is (up to a unit and a possible small rotation) a Weierstrass polynomial. This will imply that zero sets of holomorphic functions (varieties) behave a lot like zero sets of polynomials.

Theorem 1.2.2 (Weierstrass preparation theorem). *Suppose that f is holomorphic in a domain $U = U' \times \Delta$, where $0 \in U' \subset \mathbb{C}^{n-1}$ and $\Delta \subset \mathbb{C}$ is a disc centered at 0. Suppose that $z_n \mapsto f(0, z_n)$ is not identically zero on Δ and vanishes to order k at the origin.*

Then there exists a neighborhood $V \subset \mathbb{C}^n$ of 0, a unique holomorphic function u defined in V , $u(0) \neq 0$, and a unique Weierstrass polynomial p of degree k defined in V such that

$$f(z', z_n) = u(z', z_n) p(z', z_n). \quad (1.23)$$

Thus for a fixed z' we see that the zeros of f coincide with the zeros of a degree k polynomial whose coefficients depend holomorphically on z' .

In one variable, the theorem is very easy. If f is holomorphic and not identically zero and the order of vanishing at 0 is k , then $f(z) = z^k g(z)$ where $g(0) \neq 0$.

Proof. There exists a small disc $D \subset \mathbb{C}$ around zero such that $f(0, z_n)$ is not zero on $\bar{D} \setminus \{0\}$. By continuity we can find a small polydisc $V = V' \times D$ such that $\bar{V} \subset U$ and f is not zero on $V' \times \partial D$.

The one variable argument principle implies that the number of zeros (with multiplicity) of $z_n \mapsto f(z', z_n)$ in D is

$$\frac{1}{2\pi i} \int_{\partial D} \frac{\frac{\partial f}{\partial z_n}(z', z_n)}{f(z', z_n)} dz_n. \quad (1.24)$$

The number equals k for all $z' \in V'$ by continuity. We write the zeros as $\alpha_1(z'), \dots, \alpha_k(z')$ (including multiplicity). We do not give any particular order to the zeros we simply pick some order for every z' . In fact the functions α_j will not be holomorphic. What we do is write

$$p(z', z_n) = \prod_{j=1}^k (z_n - \alpha_j(z')) = z_n^k + c_{k-1}(z') z_n^{k-1} + \dots + c_0(z'). \quad (1.25)$$

As for a fixed z' , p is uniquely defined, it is clear that u and p are unique if they exist (that is, if they are holomorphic functions).

We will prove that $c_j(z')$ are holomorphic. The functions c_j are the elementary symmetric functions of the α_j 's. It is a standard (and easy) theorem in algebra that the elementary symmetric functions are polynomials in the so-called power sum functions in the α_j 's.

$$s_m(z') = \sum_{j=1}^k \alpha_j(z')^m. \quad (1.26)$$

Therefore, if we can show that s_m are holomorphic, then c_j are also holomorphic.

We use another theorem from one variable theory: If h and g are holomorphic functions on a disc D , continuous on \bar{D} , such that g has no zeros on ∂D , and $\alpha_1, \dots, \alpha_k$ are the zeros of g in D , then

$$\frac{1}{2\pi i} \int_{\partial D} h(\zeta) \frac{g'(\zeta)}{g(\zeta)} d\zeta = \sum_{j=1}^k h(\alpha_j). \quad (1.27)$$

We use the formula above and obtain that

$$s_m(z') = \sum_{j=1}^k \alpha_j(z')^m = \frac{1}{2\pi i} \int_{\partial D} z_n^m \frac{\frac{\partial f}{\partial z_n}(z', z_n)}{f(z', z_n)} dz_n. \quad (1.28)$$

We can now simply differentiate under the integral to obtain that s_m is holomorphic (justification is left as an exercise). \square

The hypotheses of the theorem are not an obstacle. If a holomorphic function f is such that $z_n \mapsto f(0, z_n)$ vanishes identically, then we can make a small linear change of coordinates L (L can be a matrix arbitrarily close to the identity) such that $f \circ L$ satisfies the hypotheses of the theorem.

Exercise 1.2.1: Prove the fact above.

Note that the order of vanishing of f at the origin is only a lower bound on the number k in the theorem. The order of vanishing for a single variable may in fact be larger. You could however again make a small linear change of coordinates so that $k = \text{ord}_0 f$.

We can think of Weierstrass preparation theorem as a generalization of the implicit function theorem. When $k = 1$ in the theorem, then we obtain the Weierstrass polynomial $z_n = c_0(z')$. That is, the zero set of f is a graph of a holomorphic function.

Example 1.2.3: A useful example to keep in mind is $f(z_1, z_2) = z_2^2 - z_1$. For all z_1 except the origin we have two zeros. For $z_1 = 0$, there is only one zero.

There is an obvious statement of the preparation theorem for germs.

Exercise 1.2.2: State and prove a germ version of the preparation theorem.

1.2.2 The division theorem

Theorem 1.2.4 (Weierstrass division theorem). *Suppose that f is holomorphic near the origin, and suppose that p is a Weierstrass polynomial of degree k in z_n . Then there exists a neighborhood N and unique holomorphic functions q and r defined in N such that r is a polynomial in z_n of degree less than k such that in N we have*

$$f = pq + r.$$

Do note that r need not be a Weierstrass polynomial; it need not be monic nor do the coefficients need to vanish at the origin. It is simply a polynomial in z_n with coefficients that are holomorphic functions of $n - 1$ variables.

Proof. The uniqueness is left as an exercise. We can assume we are working in a neighborhood $V = V' \times D$ for some small disc D such that f and p are continuous in \bar{V} and such that p is not zero on $V' \times \partial D$ and the only zeros of $p(0, z_n)$ in D are at the origin. We write

$$q(z', z_n) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z', \zeta)}{p(z', \zeta)(\zeta - z_n)} d\zeta. \quad (1.29)$$

By the assumptions on D we can see that q is holomorphic in V . Writing f using the Cauchy integral in the last variable and subtracting pq we obtain

$$r(z', z_n) = f(z', z_n) - p(z', z_n)q(z', z_n) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z', \zeta)p(z', \zeta) - f(z', \zeta)p(z', z_n)}{p(z', \zeta)(\zeta - z_n)} d\zeta. \quad (1.30)$$

We simply need to show that r is a polynomial in z_n of degree less than k . In the expression inside the integral the numerator is a polynomial of degree k in z_n that is divisible by $(\zeta - z_n)$. We perform the division and obtain a polynomial of degree $k - 1$. We can then use linearity of the integral to integrate the polynomial's coefficients. Each coefficient is a holomorphic function and we are done. Some coefficients may have integrated to zero, so we can only say that r is a polynomial of degree $k - 1$ or less. \square

Exercise 1.2.3: Prove the uniqueness part of the theorem.

1.2.3 Dependence of roots on parameters and the discriminant

Let us prove that the roots change holomorphically as long as they do not come together. Furthermore, the roots come together only on a small set; it is a zero set of a certain holomorphic function called the discriminant.

Proposition 1.2.5. *Suppose f is a holomorphic function in $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$, such that for each fixed $z' \in U'$ the function $z_n \mapsto f(z', z_n)$ has a unique zero $\alpha(z')$. Then α is holomorphic in U' .*

Proof. This is a local statement about α so we need only show that α is holomorphic near some point, which, without loss of generality, is the origin. We apply the preparation theorem and we have $f = up$ where p is a Weierstrass polynomial in $\mathcal{O}(V')[z_n]$ for some $V' \subset U'$. We have

$$p(z', z_n) = (z_n - \alpha(z'))^k = z_n^k - k\alpha(z')z_n^{k-1} + \dots \quad (1.31)$$

As the coefficients of p are holomorphic then α is holomorphic. \square

Proposition 1.2.6. *Suppose that f is holomorphic in $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$. Let $m < \infty$ be an integer such that for each $z' \in U'$, the function $z_n \mapsto f(z', z_n)$ has precisely m geometrically distinct zeros. Then there exist m holomorphic functions (defined on a perhaps smaller $V' \subset U'$) $\alpha_1(z'), \dots, \alpha_m(z')$, m integers k_1, \dots, k_m and a nonvanishing holomorphic function u such that*

$$f(z', z_n) = u(z', z_n) \prod_{j=1}^m (z_n - \alpha_j(z'))^{k_j}.$$

Exercise 1.2.4: *Prove Proposition 1.2.6 by application of Proposition 1.2.5.*

Theorem 1.2.7. *Suppose that f is holomorphic in $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$ for a connected U' . Suppose the set $f^{-1}(0)$ has no limit points on $U' \times \partial D$. For each $z' \in U'$, the function $z_n \mapsto f(z', z_n)$ has at most $m < \infty$ geometrically distinct zeros. There exists a set E such that $U' \setminus E$ is connected open and dense in U' such that $z_n \mapsto f(z', z_n)$ has exactly m zeros for $z' \in U' \setminus E$.*

Furthermore, for each point $p \in U'$ there exists a neighborhood $N \subset U'$ of p and a holomorphic function $\omega: N \rightarrow \mathbb{C}$ such that $\{z' : \omega(z') = 0\} = E \cap N$.

In particular the set of points where $z_n \mapsto f(z', z_n)$ has less than m roots is nowhere dense. The function ω is called the *discriminant function* and its zero set E is called the *discriminant set*. For example for the quadratic equation, ω is the discriminant we learned about in high school.

Proof. Proof omitted. □

1.3 The ring of germs

Let us prove some basic properties of the ring of germs of holomorphic functions.

Theorem 1.3.1. *\mathcal{O}_p is Noetherian.*

Noetherian means that for any ideal $I \subset \mathcal{O}_p$ there exist finitely many elements $f_1, \dots, f_k \in I$ such that every $g \in I$ can be written as $g = c_1 f_1 + \dots + c_k f_k$.

Proof. For simplicity we will assume that p is the origin. The proof is by induction on dimension. That ${}_1\mathcal{O}_0$ is Noetherian is a simple one variable argument that is left as an exercise.

So suppose for induction that ${}_{n-1}\mathcal{O}_0$ is Noetherian. If $I = \{0\}$ or if $I = \mathcal{O}_p$, then the assertion is obvious. Therefore, we can assume that all elements of I vanish at the origin (otherwise $I = \mathcal{O}_0$) and there exist elements that are not identically zero. Suppose that g is such an element. After perhaps a linear change of coordinates, we can assume that g is a Weierstrass polynomial in z_n by the preparation theorem.

By the division theorem, every element $f \in I$ is of the form $f = gq + r$ where r is a polynomial in z_n . We can think of r as an element of ${}_{n-1}\mathcal{O}_0[z_n]$ and furthermore $r \in I$. The set $J = I \cap {}_{n-1}\mathcal{O}_0[z_n]$ is an ideal in the ring ${}_{n-1}\mathcal{O}_0[z_n]$. By the Hilbert basis theorem, as ${}_{n-1}\mathcal{O}_0$ is Noetherian, then the polynomial ring ${}_{n-1}\mathcal{O}_0[z_n]$ is Noetherian. Thus J has finitely many generators. The generators of J together with g must generate I as any element in I is $gq + r$. □

Exercise 1.3.1: Prove that ${}_1\mathcal{O}_0$ is Noetherian, in fact it is a principal ideal domain (PID).

Theorem 1.3.2. \mathcal{O}_p is a unique factorization domain (UFD). That is, up to a multiplication by a unit, every element has a unique factorization into irreducible elements of \mathcal{O}_p .

Proof. We again assume p is the origin. We induct on the dimension. The one dimensional statement is left as an exercise. If ${}_{n-1}\mathcal{O}_0$ is a UFD, then ${}_{n-1}\mathcal{O}_0[z_n]$ is a UFD by the Gauss lemma.

Take $f \in {}_n\mathcal{O}_p$. After perhaps a linear change of coordinates $f = qW$ where q is a unit in ${}_n\mathcal{O}_0$ and W is a Weierstrass polynomial in z_n . As ${}_{n-1}\mathcal{O}_0[z_n]$ is a UFD, then W has a unique factorization in ${}_{n-1}\mathcal{O}_0[z_n]$ into $W = W_1W_2 \cdots W_k$. So $f = qW_1W_2 \cdots W_k$. That W_j are irreducible in ${}_n\mathcal{O}_p$ is left as an exercise.

Suppose that $f = \tilde{q}g_1g_2 \cdots g_m$ is another factorization. We notice that the preparation theorem applies to each g_j . Therefore write $g_j = u_j\tilde{W}_j$ for a unit u_j and a Weierstrass polynomial \tilde{W}_j . We obtain $f = u\tilde{W}_1\tilde{W}_2 \cdots \tilde{W}_m$ for a unit u . By uniqueness part of the preparation theorem we obtain $W = \tilde{W}_1\tilde{W}_2 \cdots \tilde{W}_m$. Conclusion is then obtained by noting that ${}_{n-1}\mathcal{O}_0[z_n]$ is a UFD. \square

Exercise 1.3.2: ${}_1\mathcal{O}_p$ is a unique factorization domain.

Exercise 1.3.3: Show that if an element is irreducible in ${}_{n-1}\mathcal{O}_0[z_n]$, then it is irreducible in ${}_n\mathcal{O}_0$.

1.4 Varieties

1.4.1 Definition

We can now define a variety (or a subvariety). If $f: U \rightarrow \mathbb{C}$ is a function, let $Z_f = f^{-1}(0)$ denote the zero set.

Definition 1.4.1. Let $U \subset \mathbb{C}^n$ be an open set. Let $X \subset U$ be a set such that near each point $p \in U$, there exists a neighborhood N of p and a family of holomorphic functions \mathcal{F} defined on N such that

$$N \cap X = \{z \in N : f(z) = 0 \text{ for all } f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} Z_f. \quad (1.32)$$

Then X is called a (*complex*) *variety* or a (*complex*) *subvariety* of U .

It is useful to note what happens when we replace “near each point $p \in U$ ” with “near each point $p \in X$.” We get a slightly different concept, and X is said to be a *local variety*. A local variety X is a subvariety of some neighborhood of X , but it is not necessarily closed in U . As a simple example, the set $X = \{z \in \mathbb{C}^2 : z_1 = 0, |z_2| < 1\}$ is a local variety, but not a subvariety of \mathbb{C}^2 . On the other hand X is a subvariety of the unit ball $\{z : \|z\| < 1\}$.

The family \mathcal{F} of functions can always be taken to be finite. To see this fact we first need a little bit of algebra. We will work with germs of functions. When (f, p) is a germ of a function it makes sense to talk about the germ (Z_f, p) . We can simply take the zero set of some representative and look at its germ at p .

Exercise 1.4.1: Suppose that f and g are two representatives of a germ (f, p) show that the germs (Z_f, p) and (Z_g, p) are the same.

Let

$$I_p(X) \stackrel{\text{def}}{=} \{(f, p) \in \mathcal{O}_p : (X, p) \subset (Z_f, p)\}. \quad (1.33)$$

That is, $I_p(X)$ is the set of germs of holomorphic functions vanishing on X near p . It is not hard to show that $I_p(X)$ is an ideal.

As \mathcal{O}_p is Noetherian, $I_p(X)$ must be finitely generated. Therefore, near each point p only finitely many functions are necessary to define a variety.

1.4.2 Regular points and dimension

Let us define the regular points of a variety and their dimension. If $f: U' \subset \mathbb{C}^k \rightarrow \mathbb{C}^{n-k}$ is a mapping, then by a graph of f we mean the set in $U' \times \mathbb{C}^{n-k} \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ defined by

$$\{(z, w) \in U' \times \mathbb{C}^{n-k} : w = f(z)\}. \quad (1.34)$$

Definition 1.4.2. Let $X \subset U \subset \mathbb{C}^n$ be a (complex) subvariety of U . Let $p \in X$ be a point. If there exists a (complex) linear change of coordinates such that near p the set X can be written as a graph of a holomorphic mapping $f: U' \subset \mathbb{C}^k \rightarrow \mathbb{C}^{n-k}$ (for some $k \in \mathbb{N}_0$), then p is a *regular point* (or *simple point*) of X and the *dimension* of X at p is k or $\dim_p X = k$.

The set of regular points of X will be denoted by X_{reg} . Any point that is not regular is *singular*. The set of singular points of X is denoted by X_{sing} .

So $p \in X$ is a regular point if after perhaps a linear change of coordinates $X \cap (U' \times \mathbb{C}^{n-k}) = \{(z, w) \in U' \times \mathbb{C}^{n-k} : w = f(z)\}$ where $p \in X \cap U' \times \mathbb{C}^{n-k}$. In other words, p is a regular point of X (of dimension k) if near p X is a k -dimensional complex manifold near p . Note that an isolated point of the variety is a regular point of dimension 0. We have the following theorem, which we state without proof.

Theorem 1.4.3. Let $U \subset \mathbb{C}^n$ be open and connected and let $X \subset U$ be a subvariety, then the set of regular points X_{reg} is open and dense in X . In fact X_{sing} is a subvariety of X .

We will prove the following simpler version to get a feeling for why the theorem is true.

Proposition 1.4.4. Let $U \subset \mathbb{C}^n$ be open and connected. Let $f: U \rightarrow \mathbb{C}$ be holomorphic. Then $(Z_f)_{reg}$ is open and dense in Z_f .

In fact, $(Z_f)_{sing}$ is contained in the zero set of some holomorphic function that is not zero on any open set of Z_f .

Proof. Without loss of generality, we need only prove this locally near the origin. After a possibly linear change of coordinates we can apply the preparation theorem to find that near the origin, Z_f is the zero set of a Weierstrass polynomial $p(z', z_n)$ defined on some $U' \times \mathbb{C}$. Let E be the discriminant set for p . Above each point in $U' \setminus E$ the set is union of m distinct graphs of holomorphic functions by Proposition 1.2.6. As before, above each point of E we only have finitely many points of Z_f , and the conclusion follows. \square

A related result we will also not prove is the following.

Theorem 1.4.5. *Let $U \subset \mathbb{C}^n$ be open and connected and let $X \subset U$ be a subvariety, then the X is a locally finite union of manifolds.*

We can also define dimension at a singular point. We stated above (without proof) that the set of regular points of a complex subvariety is open and dense in the subvariety. Thus, a variety is a manifold at most points. This of course means that the following definition does make sense.

Definition 1.4.6. Let $X \subset U \subset \mathbb{C}^n$ be a (complex) subvariety of U . Let $p \in X$ be a point. We define the *dimension* of X at p to be

$$\dim_p X \stackrel{\text{def}}{=} \max\{k \in \mathbb{N}_0 : \forall \text{ neighborhoods } N \text{ of } p, \exists q \in N \cap X_{\text{reg}} \text{ with } \dim_q X = k\}. \quad (1.35)$$

We also define the *dimension* of the whole variety as

$$\dim X \stackrel{\text{def}}{=} \max_{p \in X} \dim_p X. \quad (1.36)$$

Finally we will also sometimes use the word *codimension*. If X is a subvariety of an open set of \mathbb{C}^n and has dimension k , then we say that X is of *codimension* $n - k$.

1.4.3 Hypervarieties

Codimension 1 subvarieties are particularly nice. Sometimes codimension 1 subvarieties are called *hypervarieties*. We state the following theorem without proof.

Theorem 1.4.7. *Let $U \subset \mathbb{C}^n$ be open and connected. Let $f: U \rightarrow \mathbb{C}$ be holomorphic. Then Z_f is either empty, codimension 1, or $Z_f = U$.*

Conversely, if (X, p) is a germ of a codimension 1 variety, then there is a germ holomorphic function f at p such that $(Z_f, p) = (X, p)$.

Example 1.4.8: Such simple statements are not true in general. It is simply not true that if a dimension of a variety in \mathbb{C}^n is $n - k$, then there are k holomorphic functions that “cut it out.” The set defined by

$$\text{rank} \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{bmatrix} < 2 \quad (1.37)$$

is a 4 dimensional variety in \mathbb{C}^6 and the defining equations are $z_1 z_5 - z_2 z_4 = 0$, $z_1 z_6 - z_3 z_4 = 0$, and $z_2 z_6 - z_3 z_5 = 0$. The unique singular point is the origin and there exist no 2 holomorphic functions that define this variety. In more technical language, the variety is not a *complete intersection*.

1.4.4 Irreducible varieties

Definition 1.4.9. A germ of a complex variety $(X, p) \subset (\mathbb{C}^n, p)$ is said to be *reducible* at p if there exist two germs (X_1, p) and (X_2, p) with $(X_1, p) \not\subset (X_2, p)$ and $(X_2, p) \not\subset (X_1, p)$ such that $(X, p) = (X_1, p) \cup (X_2, p)$. Otherwise, the germ (X, p) is *irreducible* at p .

Similarly globally, a subvariety $X \subset U$ is *reducible* in U if there exist two subvarieties X_1 and X_2 of U with $X_1 \not\subset X_2$ and $X_2 \not\subset X_1$ such that $X = X_1 \cup X_2$. Else the germ (X, p) is *irreducible* in U .

It turns out that for the local notion, we simply need to check that there does not exist a germ $(Y, p) \subset (X, p)$ $(Y, p) \neq (X, p)$ and $\dim_p X = \dim_p Y$. We state without proof that if a complex variety is irreducible, then it has the same dimension at all points, and furthermore the set of regular points is connected. In fact, much more is true and we give the following theorem without proof (we leave the special case of codimension 1 as an exercise).

Theorem 1.4.10 (Local parametrization theorem). *Let $(V, 0)$ be an irreducible germ of a complex variety of dimension k in \mathbb{C}^n . Let V denote a representative of the germ. Then after a linear change of coordinates, we let $\pi: \mathbb{C}^n \rightarrow \mathbb{C}^k$ be the projection onto the first k components, and obtain that there exists a neighborhood $U \subset \mathbb{C}^n$ of the origin, and a proper subvariety $E \subset \pi(U)$ such that*

- (i) $V' = V \cap U \setminus \pi^{-1}(E)$ is a connected k -dimensional complex manifold that is dense in $V \cap U$.
- (ii) $\pi: V' \rightarrow \pi(U) \setminus E$ is an m -sheeted covering map for some integer m .
- (iii) $\pi: V \cap U \rightarrow \pi(U)$ is a proper mapping.

An m -sheeted covering map in this case will be a local biholomorphism that is an m -to-1 map.

Exercise 1.4.2: Use Theorem 1.2.7 to prove the parametrization theorem if V is defined by the vanishing of a single holomorphic function. Do not prove that V' is connected.

For varieties defined by the vanishing of a single holomorphic function, we see that by the UFD property of ${}_n\mathcal{O}_p$ we can find finitely many so-called *irreducible components*. We state without proof that this property holds for all varieties.

Example 1.4.11: Note that the local and global behavior are different. For example

$$z_2^2 = z_1(z_1 - 1)^2 \tag{1.38}$$

is irreducible in \mathbb{C}^2 (the regular points are connected) but locally at the point $(1, 0)$ it is reducible. See Figure 1.1 for the plot in two real dimensions.

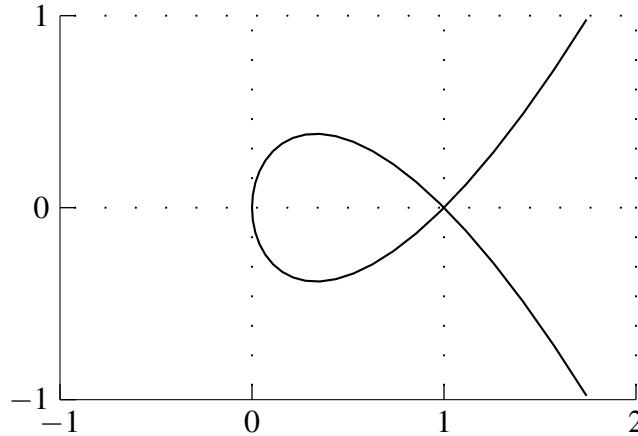


Figure 1.1: Locally reducible but globally irreducible curve.

1.4.5 Curves

Complex varieties of dimension one will play a crucial role. Any one dimensional variety is locally an image of a disc under a holomorphic one-to-one mapping. First, let us define what we mean by an analytic curve.

Definition 1.4.12. Let $X \subset \mathbb{C}^n$ be a set such that near every point $p \in X$, there exists a holomorphic map $f: \mathbb{D} \rightarrow \mathbb{C}^n$ ($\mathbb{D} \subset \mathbb{C}$ is the unit disc) that is one-to-one and such that for some open neighborhood N of p we have $f(\mathbb{D}) = X \cap N$. Then X is an *analytic curve*.

For example image of $z \mapsto (z^2, z^3)$ is the variety (analytic curve) defined by $z_1^3 - z_2^2 = 0$ in \mathbb{C}^2 . We state without proof the following theorem.

Theorem 1.4.13 (Puiseux normalization). *Let $(X, p) \subset (\mathbb{C}^n, p)$ be an irreducible germ of a dimension one subvariety. Then X is a germ of an analytic curve.*

In particular, if $X \subset \mathbb{C}^2$ is the zero set of some function f irreducible in \mathcal{O}_0 and $f(0, z)$ is not identically zero, then there exists a holomorphic function g such that $f(z^k, g(z)) = 0$ ($z \in \mathbb{C}$). This is the statement that is generally referred to as the Puiseux theorem.

A consequence of this theorem is that one dimensional complex varieties are topological manifolds, even at the singular points. This is not true for higher dimensions. One dimensional varieties are simply very nice.

Because of this theorem, whenever we are looking for germs of one dimensional complex varieties, we will generally simply parametrize them. For varieties of larger dimensions we can always find enough curves through any point, and then we can parametrize those.

Exercise 1.4.3: *Using the local parametrization theorem, prove that if (X, p) is an irreducible germ of a complex variety of dimension greater than 1, then there exists a neighborhood U of p and a closed subvariety $X \subset U$ (whose germ at p is (X, p)), such that for every $q \in X$ there exists an irreducible subvariety $Y \subset X$ of dimension 1 such that $p \in Y$ and $q \in Y$.*

Chapter 2

CR Geometry

2.1 Proper maps and the inequivalence of ball and polydisc

2.1.1 Proper maps

Definition 2.1.1. Let X and Y be topological spaces. A continuous mapping $f: X \rightarrow Y$ is said to be *proper* if $f^{-1}(K) \subset X$ is compact whenever $K \subset Y$ is compact.

Being proper is a purely topological notion. It can be thought of as the first step towards having a continuous inverse. We know that a continuous map pushes forward compact sets but inverse images of compact sets could be only closed. The inverse image of compact sets being compact is a necessary, but by far not sufficient condition for having a continuous inverse.

One way to understand proper maps is by looking at compactifications. Let X^∞ and Y^∞ be the one point compactifications of X and Y and define $f^\infty: X^\infty \rightarrow Y^\infty$ by $f^\infty(\infty) = \infty$ and $f^\infty|_X = f$.

Exercise 2.1.1: Show that f is proper if and only if f^∞ is continuous (Note that f is continuous to start with).

It seems to therefore make sense that proper maps that extend to the boundaries of domains take boundary to boundary. In particular we have the following lemma.

Lemma 2.1.2. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be bounded domains and let $f: U \rightarrow V$ be continuous. Then f is proper if and only if for every sequence $\{z_k\}$ in U such that $z_k \rightarrow p \in \partial U$, the set of limit points of $\{f(z_k)\}$ lies in ∂V .

Proof. First suppose that f is proper. Take a sequence $\{z_k\}$ in U such that $z_k \rightarrow p \in \partial U$. Then take any convergent subsequence $f(z_{k_j})$ of $\{f(z_k)\}$ converging to some $q \in \bar{V}$. Let $E = \{f(z_{k_j})\}$. The inverse image $f^{-1}(\bar{E})$ is not compact in U and hence \bar{E} is not compact in V and thus $q \notin V$.

Conversely suppose that for every sequence $\{z_k\}$ in U such that $z_k \rightarrow p \in \partial U$, the set of limit points of $\{f(z_k)\}$ lies in ∂V . Take a closed set $E \subset V$ and look at $f^{-1}(E)$. If $f^{-1}(E)$ is not compact

in U , then there exists a sequence $\{z_k\}$ in $f^{-1}(E)$ such that $z_k \rightarrow p \in \partial U$. But then the limit points of $\{f(z_k)\}$ are in ∂V and hence E has limit points in ∂V and thus is not compact. \square

Corollary 2.1.3. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be bounded domains. Let $f: \bar{U} \rightarrow \bar{V}$ be a continuous map such that $f(U) \subset V$. Then $f|_U: U \rightarrow V$ is a proper map if and only if $f(\partial U) \subset \partial V$.*

2.1.2 Inequivalence of ball and polydisc

First let us consider the boundaries of the unit bidisc $\mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ and the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$. We notice that the boundary of the unit bidisc contains analytic curves $\{p\} \times \mathbb{D}$ and $\mathbb{D} \times \{p\}$ for $p \in \partial \mathbb{D}$. On the other hand we have the following proposition.

Proposition 2.1.4. *The unit sphere $S^{2n-1} = \partial \mathbb{B}_n \subset \mathbb{C}^n$ contains no analytic curves.*

Proof. Suppose we have a holomorphic function $g: U \subset \mathbb{C} \rightarrow \mathbb{C}^n$ such that the image of g is inside the unit sphere. In other words

$$\|g(z)\| = 1 \quad (2.1)$$

for all $z \in U$. Without loss of generality (after composing with a unitary matrix) we can assume that $g(0) = (1, 0, 0, \dots, 0)$. We look at the first component and notice that $g_1(0) = 1$ and $|g_1(z)| < 1$ for all z such that $g_1(z) \neq 1$. By the maximum principle $g_1(z) = 1$ for all z and therefore $g(z) = g(0)$ for all z . \square

The fact that the sphere contains no nontrivial varieties (no curves) is the most important geometric distinction between the boundary of the polydisc and the sphere.

Theorem 2.1.5 (Rothstein 1935). *There exists no proper mapping of the unit bidisc $\Delta = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ to the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$.*

Proof. Suppose that we have a proper holomorphic map $f: \Delta \rightarrow \mathbb{B}_2$. Fix some $e^{i\theta}$ in the boundary of the disc \mathbb{D} . Take a sequence $w_k \in \mathbb{D}$ such that $w_k \rightarrow e^{i\theta}$. The functions $g_k(\zeta) = f(\zeta, w_k)$ map the unit disc into \mathbb{B}_2 . By the standard Montel's theorem, by passing to a subsequence we can assume that the sequence of functions converges to a limit $g: \mathbb{D} \rightarrow \bar{\mathbb{B}}_2$. As $(\zeta, w_k) \rightarrow \partial \Delta$, then by Lemma 2.1.2 we have that $g(\mathbb{D}) \subset \partial \mathbb{B}_2$ and hence g must be constant.

As the functions g'_k converge to g' , then for every fixed $\zeta \in \mathbb{D}$, $\frac{\partial f}{\partial z_1}(\zeta, w_k) \rightarrow 0$. Notice that this was true for all $e^{i\theta}$ and a subsequence of an arbitrary sequence $w_k \rightarrow e^{i\theta}$. We can apply the maximum principle to find that $\frac{\partial f}{\partial z_1} \equiv 0$. By symmetry $\frac{\partial f}{\partial z_2} \equiv 0$. Therefore f would be a constant, which is a contradiction as f was proper. \square

As a biholomorphic mapping is proper, we see that the unit bidisc is not biholomorphically equivalent to the unit ball in \mathbb{C}^2 . This fact was first proved by Poincaré by computing the automorphism groups of Δ and \mathbb{B}_2 , although the proof assumed that the maps extended to the boundary. The first real proof was by H. Cartan in 1931, though popularly the theorem is attributed to Poincaré.

We saw that the reason why there is not even a proper mapping is the fact that the boundary of the polydisc contained analytic curves, while the sphere did not. It will be therefore useful to know when does the boundary of a domain contain complex varieties.

Similar proof extends to higher dimensions as well. In fact, it is not hard to prove the following theorem.

Theorem 2.1.6. *Let $U = U' \times U'' \subset \mathbb{C}^n \times \mathbb{C}^k$ and $V \subset \mathbb{C}^m$ be bounded domains such that ∂V contains no nontrivial complex analytic subvarieties. Then there exist no proper holomorphic mappings $f: U \rightarrow V$.*

Note the difference with one variable theory. In one variable, any two simply connected domains are biholomorphically equivalent. In two or more variables this is no longer the case, even for the two most simple examples. The geometry of the boundaries makes a difference, not just the topology of the domains.

2.2 Polarization

We prove an immensely useful result for CR geometry. As a consequence, we can treat z and \bar{z} as separate variables. First we define the complex conjugate for a domain $U \subset \mathbb{C}^n$

$$U^* \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n : \bar{z} \in U\}. \quad (2.2)$$

Proposition 2.2.1 (Polarization). *Let $U \subset \mathbb{C}^n$ be open and connected. Let $f: U \times U^* \rightarrow \mathbb{C}$ be holomorphic such that for all $z \in U$ we have*

$$f(z, \bar{z}) = 0, \quad (2.3)$$

then $f(z, w) = 0$ for all $(z, w) \in U \times U^*$

Proof. Without loss of generality, assume that U is some small neighborhood of the origin where the power series of f converges. Write down the power series for f at the origin in multi-index notation.

$$f(z, w) = \sum_{\alpha\beta} c_{\alpha\beta} z^\alpha w^\beta. \quad (2.4)$$

Let δ_j^k be the Kronecker delta (that is, $\delta_j^k = 0$ if $j \neq k$ and $\delta_j^j = 1$). Elementary calculation shows that $\frac{\partial}{\partial z_j} z_k = \delta_j^k$, $\frac{\partial}{\partial z_j} \bar{z}_k = 0$, $\frac{\partial}{\partial \bar{z}_j} z_k = 0$, $\frac{\partial}{\partial \bar{z}_j} \bar{z}_k = \delta_j^k$. Therefore for any two multi-indexes α and β by differentiating the power series and evaluating at $z = 0$ we obtain

$$\left. \frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \bar{z}^\beta} \right|_{z=0} f(z, \bar{z}) = \left. \frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \bar{z}^\beta} \right|_{z=0} \sum_{\mu\nu} c_{\mu\nu} z^\mu \bar{z}^\nu = \alpha! \beta! c_{\alpha\beta}. \quad (2.5)$$

Since $f(z, \bar{z})$ is identically zero, we get that $c_{\alpha\beta} = 0$. Hence $f \equiv 0$. \square

Thus if we can write a function $g(x, y)$ defined on \mathbb{C}^n (where $x + iy = z \in \mathbb{C}^n$) as $f(z, \bar{z})$ for a holomorphic function f in $2n$ variables, then we can treat \bar{z} as a separate variable, as any real-analytic identity involving z and \bar{z} extends to a holomorphic identity where \bar{z} is a separate variable.

Treating \bar{z} as a separate variable is sometimes also called *complexification*.

2.3 Hermitian forms for polynomials

2.3.1 Polynomials as Hermitian forms

We start with relating real polynomials and Hermitian forms. We will build up to real-analytic functions later. The results of this section in the context of complex analysis are due to D'Angelo.

Let $x + iy = z \in \mathbb{C}^n$ be the coordinates. Suppose we have a real-valued polynomial $p(x, y)$. We will define the polynomial r on $\mathbb{C}^n \times \mathbb{C}^n$ by

$$r(z, \bar{w}) = p\left(\frac{z + \bar{w}}{2}, \frac{z - \bar{w}}{2i}\right). \quad (2.6)$$

Now notice that $r(z, \bar{z}) = p(x, y)$. We have simply written p in terms of z and \bar{z} instead of x and y . We will note that it will be advantageous to consider z and \bar{z} as separate variables.

As p is real-valued we have that $r(z, \bar{z}) = \overline{r(z, \bar{z})}$. By polarization (last section), we note that for all $w \in \mathbb{C}^n$ we have

$$r(z, \bar{w}) = \overline{r(w, \bar{z})}. \quad (2.7)$$

A polynomial that satisfies (2.7) we will call a *Hermitian symmetric polynomial*. The reason for this notation is that as we will see, the polynomial is an honest Hermitian form on the space of holomorphic polynomials.

Write r in multi-index notation

$$r(z, \bar{w}) = \sum_{\alpha\beta} c_{\alpha\beta} z^\alpha \bar{w}^\beta.$$

We order all the multi-indexes in some way. This gives us a certain ordering of the monomials in z (and in \bar{w}). For example, $1, z_1, z_2, \dots, z_n, z_1^2, z_1 z_2, \dots$. After ordering it makes sense to write down the matrix $C = [c_{\alpha\beta}]$. We call C the *matrix of coefficients*. For convenience, let the rows be denoted by the β and the columns by α . We will write $X^* = \overline{X^T}$ for the conjugate transpose of X .

As r is Hermitian symmetric, C is Hermitian symmetric as it is easy to check that the Hermitian symmetry of r implies that $c_{\alpha\beta} = \overline{c_{\beta\alpha}}$. Note that we only really need to write down the pure holomorphic or pure antiholomorphic monomials that appear in the polynomial.

Example 2.3.1: Take $r(z, \bar{z}) = -iz_1 \bar{z}_2 + iz_2 \bar{z}_1 + 3|z_2^2|^2$. Then let us order the monomials that appear as (z_1, z_2, z_2^2) . The 3×3 matrix C is then

$$C = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \text{so that} \quad r(z, \bar{z}) = \begin{bmatrix} z_1 \\ z_2 \\ z_2^2 \end{bmatrix}^* C \begin{bmatrix} z_1 \\ z_2 \\ z_2^2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_2^2 \end{bmatrix}^* \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_2^2 \end{bmatrix}. \quad (2.8)$$

We can see that a real-valued polynomial on \mathbb{C}^n can be regarded as a Hermitian form on the space of holomorphic polynomials. When we use the monomial basis for that space, we recover the matrix C . Notice that once we fix the basis and the ordering, then C is uniquely determined.

2.3.2 Decomposition as difference of squared norms

Suppose that C has a positive and b negative eigenvalues. We will call the pair (a, b) the *signature pair* of r . We will prove the following proposition.

Proposition 2.3.2. *Let $r(z, \bar{z})$ be a Hermitian symmetric polynomial with signature pair (a, b) . Then there exist two holomorphic polynomial mappings $f: \mathbb{C}^n \rightarrow \mathbb{C}^a$ and $g: \mathbb{C}^n \rightarrow \mathbb{C}^b$, such that*

$$r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2. \quad (2.9)$$

Proof. Take v_1, \dots, v_{a+b} to be an orthonormal set of eigenvectors of the matrix of coefficients corresponding to the nonzero eigenvalues. Let v_1, \dots, v_a be the eigenvectors corresponding to the positive eigenvalues $\lambda_1, \dots, \lambda_a$, and let v_{a+1}, \dots, v_{a+b} be the eigenvectors corresponding to the negative eigenvalues $\lambda_{a+1}, \dots, \lambda_{a+b}$.

We have

$$C = \sum_{j=1}^{a+b} \lambda_j v_j v_j^* \quad (2.10)$$

where the star denotes conjugate transpose.

Exercise 2.3.1: *Prove the identity above.*

Let \mathcal{Z} denote the column vector of monomials sorted as above and write $w_j = \sqrt{|\lambda_j|} v_j$. Then

$$\begin{aligned} C &= \sum_{j=1}^a \lambda_j v_j v_j^* - \sum_{j=a+1}^{a+b} (-\lambda_j) v_j v_j^* \\ &= \sum_{j=1}^a \sqrt{\lambda_j} v_j \sqrt{\lambda_j} v_j^* - \sum_{j=a+1}^{a+b} \sqrt{-\lambda_j} v_j \sqrt{-\lambda_j} v_j^* \\ &= \sum_{j=1}^a w_j w_j^* - \sum_{j=a+1}^{a+b} w_j w_j^* \end{aligned} \quad (2.11)$$

For $j = 1, \dots, a$ define $f_j(z) = w_j^* \mathcal{Z}$. For $j = 1, \dots, b$ define $g_j(z) = w_{a+j}^* \mathcal{Z}$. Then notice that

$|f_j(z)|^2 = \mathcal{L}^* w_j w_j^* \mathcal{L}$ and $|g_j(z)|^2 = \mathcal{L}^* w_{a+j} w_{a+j}^* \mathcal{L}$. Therefore

$$\begin{aligned}
r(z, \bar{z}) &= \langle C\mathcal{L}, \mathcal{L} \rangle = \mathcal{L}^* C\mathcal{L} \\
&= \sum_{j=1}^a \mathcal{L}^* w_j w_j^* \mathcal{L} - \sum_{j=a+1}^{a+b} \mathcal{L}^* w_j w_j^* \mathcal{L} \\
&= \sum_{j=1}^a |f_j(z)|^2 - \sum_{j=1}^b |g_j(z)|^2 \\
&= \|f(z)\|^2 - \|g(z)\|^2.
\end{aligned} \tag{2.12}$$

□

Example 2.3.3: Let us continue the previous example $r(z, \bar{z}) = -iz_1 \bar{z}_2 + iz_2 \bar{z}_1 + 3|z_2^2|^2$. We have $\mathcal{L} = (z_1, z_2, z_2^2)^t$. The eigenvalues are 3, 1, and -1 . The corresponding set of orthonormal eigenvectors are

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \quad v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \tag{2.13}$$

and $w_1 = \sqrt{3}v_1$, $w_2 = v_2$ and $w_3 = v_3$. Thus

$$f_1(z) = \sqrt{3}z_2^2, \quad f_2(z) = \frac{z_1 + iz_2}{\sqrt{2}}, \quad g_1(z) = \frac{z_1 - iz_2}{\sqrt{2}}, \tag{2.14}$$

and

$$r(z, \bar{z}) = \left| \sqrt{3}z_2^2 \right|^2 + \left| \frac{z_1 + iz_2}{\sqrt{2}} \right|^2 - \left| \frac{z_1 - iz_2}{\sqrt{2}} \right|^2. \tag{2.15}$$

2.3.3 Pure parts

Note that in the decomposition given above, it is not necessary for all the functions f_j and g_j to vanish at the origin, even if $r(0, 0) = 0$. For example, when $r(z, \bar{z}) = z_1 + \bar{z}_1$, then we can write

$$r(z, \bar{z}) = \left| \frac{z_1 + 1}{\sqrt{2}} \right|^2 - \left| \frac{z_1 - 1}{\sqrt{2}} \right|^2. \tag{2.16}$$

Sometimes it is useful to have all the functions vanish at the origin. Notice that it is precisely the terms in the polynomial that are purely holomorphic (depend only on z) or purely antiholomorphic (depend only on \bar{z}) that make trouble. If we exclude those terms, then none of the eigenvectors will involve the constant term and therefore, the resulting functions will vanish at the origin. Furthermore, by writing all the polynomials as polynomials in $z - p$, we can make sure that if $r(p, \bar{p}) = 0$, then all the functions in the decomposition vanish at p .

Proposition 2.3.4. *Let $r(z, \bar{z})$ be a Hermitian symmetric polynomial and $p \in \mathbb{C}^n$ such that $r(p, \bar{p}) = 0$. Then there exists a holomorphic polynomial $h(z)$, and two holomorphic polynomial mappings $f(z)$, $g(z)$, all vanishing at p such that*

$$r(z, \bar{z}) = 2\operatorname{Re}h(z) + \|f(z)\|^2 - \|g(z)\|^2. \quad (2.17)$$

We allow the possibility that any of h , f , or g may in fact be missing (i.e. be identically zero) for certain r .

Proof. Without loss of generality, assume that $p = 0$. Take $h(z) = r(z, 0)$, that is, $h(z)$ is the polynomial consisting of all the holomorphic terms in $r(z, \bar{z})$. Then we note that all the antiholomorphic terms in r are in fact $\overline{h(z)}$ as r is Hermitian symmetric. Note that $2\operatorname{Re}h(z) = h(z) + \overline{h(z)}$. As $r(0, 0) = 0$, there is no constant term. Therefore $h(0) = 0$.

Now we notice that the coefficient matrix for the Hermitian symmetric polynomial $r(z, \bar{z}) - 2\operatorname{Re}h(z)$ has no entries in the row and column corresponding to the constant monomial. Therefore, all the eigenvectors corresponding to nonzero eigenvalues only have nonzero entries for nonconstant monomials. Therefore the functions $w_j^* \mathcal{L}$ must vanish at the origin. \square

2.4 Complex varieties in real-algebraic hypersurfaces

A *real-algebraic hypersurface* is a real submanifold M of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ of real dimension $2n - 1$ that is defined by

$$M = \{z \in \mathbb{C}^n : r(z, \bar{z}) = 0\}$$

for some real-valued (Hermitian symmetric) polynomial r . For simplicity, we will assume that M is really a submanifold and is not singular.

Suppose that $p \in M$ is a point such that through p there is a positive dimensional complex variety contained in M . Without loss of generality suppose that this variety is an analytic curve. That is, there exists a nonconstant holomorphic function $\varphi: \mathbb{D} \rightarrow \mathbb{C}^n$ such that $\varphi(\mathbb{D}) \subset M$, and $\varphi(0) = p$. That is, we have that for all $t \in \mathbb{D}$ we have

$$r(\varphi(t), \overline{\varphi(t)}) = 0. \quad (2.18)$$

As a useful notation, denote by $\bar{\varphi}$ the holomorphic function that is obtained by taking the complex conjugate of all the coefficients in the power series of the function φ . Then $\overline{\varphi(t)} = \bar{\varphi}(\bar{t})$.

In the equation $0 = r(\varphi(t), \overline{\varphi(t)}) = r(\varphi(t), \bar{\varphi}(\bar{t}))$, we can treat \bar{t} as a separate variable. Now decompose r as in Proposition 2.3.4

$$r(z, \bar{z}) = 2\operatorname{Re}h(z) + \|f(z)\|^2 - \|g(z)\|^2. \quad (2.19)$$

Plug in $z = \varphi(t)$ and set $\bar{t} = 0$ and then notice that $\varphi(0) = p$. In the following the dot \cdot denotes the simple symmetric (not Hermitian) dot product: $\xi \cdot \zeta = \xi_1 \zeta_1 + \cdots + \xi_n \zeta_n$.

$$\begin{aligned} 0 = r(\varphi(t), \bar{\varphi}(0)) &= h(\varphi(t)) + \bar{h}(\bar{\varphi}(0)) + f(z) \cdot \bar{f}(\bar{\varphi}(0)) - g(z) \cdot \bar{g}(\bar{\varphi}(0)) \\ &= h(\varphi(t)) + \overline{h(p)} + f(z) \cdot \overline{f(p)} - g(z) \cdot \overline{g(p)} \\ &= h(\varphi(t)). \end{aligned} \tag{2.20}$$

Therefore h vanishes on the image of φ .

Note that $\{z \in \mathbb{C}^n : h(z) = r(z, p) = 0\}$ defines the so-called *Segre variety* Σ_p for M at p . We will talk more about this useful variety later on. For now, let us simply note that we have proved the following proposition.

Proposition 2.4.1. *Let $M \subset \mathbb{C}^n$ be a real-algebraic hypersurface. Let (X, p) be a germ of a complex variety such that $(X, p) \subset (M, p)$. Then $(X, p) \subset (\Sigma_p, p)$.*

Example 2.4.2: The *Lewy hypersurface* $M \subset \mathbb{C}^n$ defined by

$$\text{Im } z_n = \sum_{j=1}^{n-1} |z_j|^2 \tag{2.21}$$

contains no nontrivial complex varieties through the origin. We write $r(z, \bar{z}) = \text{Im } z_n - \sum_{j=1}^{n-1} |z_j|^2$. Then we look at $r(z, 0) = z_n/2i$. Thus $\Sigma_0 = \{z : z_n = 0\}$. If we plug $z_n = 0$ this into the defining equation of M , then we obtain $0 = \sum_{j=1}^{n-1} |z_j|^2$, and hence $M \cap \Sigma_0 = \{0\}$.

The Lewy hypersurface is in fact locally biholomorphically equivalent to a sphere, so we have seen this statement before.

On the other hand, just because Σ_p intersects M in a nontrivial way, it doesn't mean that M contains complex varieties. All we do know is that any complex variety contained in M is contained in $M \cap \Sigma_p$.

Theorem 2.4.3 (D'Angelo). *Let $M \subset \mathbb{C}^n$ be an algebraic hypersurface defined by $r(z, \bar{z}) = 0$. Let $p \in M$ be a point and let*

$$r(z, \bar{z}) = 2\text{Re } h(z) + \|f(z)\|^2 - \|g(z)\|^2, \tag{2.22}$$

where h , f , and g vanish at p . By adding zero components assume that target dimensions of f and g are the same. Then M contains a germ of complex variety at p if and only if there exists a unitary matrix U such that the variety $V(h, f - Ug)$ (i.e. the variety defined by $h = 0$ and $f = Ug$) is nontrivial.

In fact, $V = V(h, f - Ug) \subset M$ for every unitary U , and if (X, p) is an irreducible germ of a complex variety such that $(X, p) \subset (M, p)$ then there exists a unitary U such that $(X, p) \subset (V, p)$.

Proof. We start with the easy direction. Suppose that there exists a unitary matrix U such that $V(h, f - Ug)$ is nontrivial. For a unitary matrix U we have that $\|Ug(z)\|^2 = \|g(z)\|^2$. So if z is such that $f(z) = Ug(z)$ and $h(z) = 0$, then $2\operatorname{Re}h(z) + \|f(z)\|^2 - \|g(z)\|^2 = 0$. Thus $V(h, f - Ug) \subset M$.

For the other direction, suppose that there is an irreducible germ of a complex variety through p contained in M . Take a representative X of this variety. As we have seen, for any nonconstant holomorphic mapping $\varphi: \mathbb{D} \rightarrow X \subset M$ with $\varphi(0) = p$ we have that $h(\varphi(t)) = 0$ for all $t \in \mathbb{D}$. Therefore, for all $t \in \mathbb{D}$ we have $\|f(\varphi(t))\|^2 = \|g(\varphi(t))\|^2$. Since this is true for any curve that lies in $X \subset M$, we know that there exists a regular point $\xi \in X$, such that $\|f(z)\|^2 = \|g(z)\|^2$ for all $z \in X$ near ξ . So parametrize a neighborhood $N \subset X$ of ξ via a mapping $\Phi: \Delta' \rightarrow N$ for some polydisc Δ' of the same dimension as X . Then for $z \in \Delta'$

$$\|f(\Phi(z))\|^2 = \|g(\Phi(z))\|^2. \quad (2.23)$$

Note that for a mapping $F: \mathbb{C}^n \rightarrow \mathbb{C}^m$ and a unitary matrix U we have $\|F(z)\|^2 = \|UF(z)\|^2$. What's interesting is that the converse holds as well. We need the following lemma.

Lemma 2.4.4. *Let $V \subset \mathbb{C}^n$ be a domain and let $F: V \rightarrow \mathbb{C}^m$ and $G: V \rightarrow \mathbb{C}^m$ be holomorphic mappings such that for all $z \in V$ we have*

$$\|F(z)\|^2 = \|G(z)\|^2, \quad (2.24)$$

then there exists a unitary U such that $F(z) = UG(z)$.

Proof. Without loss of generality, assume that V is a neighborhood of the origin where the power series for both F and G converges. Denote by F the $m \times \infty$ matrix of coefficients of the function $F(z)$ and similarly with G . Let \mathcal{Z} be an infinite column vector of all the monomials in z such that $F(z) = F\mathcal{Z}$ and $G(z) = G\mathcal{Z}$. We then see that $\|F(z)\|^2 = \mathcal{Z}^* F^* F \mathcal{Z}$ and $\|G(z)\|^2 = \mathcal{Z}^* G^* G \mathcal{Z}$. By polarization and the fact that $\|F(z)\|^2 = \|G(z)\|^2$ we have that $F^* F = G^* G$.

Obviously $F^* F$ and $G^* G$ have the same rank, so we take unitary matrices U' and U'' such that $U'F = \begin{bmatrix} F' \\ 0 \end{bmatrix}$ and $U''G = \begin{bmatrix} G' \\ 0 \end{bmatrix}$ and F' and G' are of full rank (rank F' is equal to the number of rows of F'). Thus without loss of generality we can assume that F and G are of full rank.

If we wish we can work with finite matrices. Let F_k denote the first k columns of F . Then we see that the leading principal $k \times k$ submatrix of $F^* F$ is $F_k^* F_k$. As $F_k^* F_k = G_k^* G_k$, then by standard linear algebra there exists a unitary matrix U_k such that $F_k = U_k G_k$. Since both F_k and G_k are of full rank, then there exists a K such that for $k \geq K$, F_k and G_k are of full rank. It is then easy to see that for all $k, \ell \geq K$ we must have $U_\ell = U_k$. Hence taking $U = U_K$ gives the theorem. \square

Armed with the lemma, we now look at

$$\|f(\Phi(z))\|^2 = \|g(\Phi(z))\|^2. \quad (2.25)$$

We have assumed that f and g have the same number of components by adding zero components if necessary. Applying the lemma we have that there exists a unitary matrix U such that for all z

$$f(\Phi(z)) = Ug(\Phi(z)). \quad (2.26)$$

And we are essentially done. We have that $\Phi(\Delta') \subset V(h, f - Ug)$, and so $X \subset V(h, f - Ug)$ as X is irreducible. \square

Before we move on, let us remark about a simple scenario for a certain important class of hypersurfaces. Suppose that we can write the defining equation for M as

$$2\operatorname{Re}h(z) + \|f(z)\|^2 = 0, \quad (2.27)$$

where $h(p) = 0$ and $f(p) = 0$. Then M contains a germ of a nontrivial complex variety through p if and only if $V(h, f)$ is nontrivial.

Example 2.4.5: The algebraic hypersurface $M \subset \mathbb{C}^3$ defined by

$$0 = 2\operatorname{Re}z_3 + |z_1|^2 - |z_2|^2 \quad (2.28)$$

contains many complex lines through the origin. In particular any line of the form $\{z : z_3 = 0, z_1 = e^{i\theta}z_2\}$ for some fixed θ . No other complex varieties exist. Furthermore there is not a single unique complex variety sitting in M . The union of all the lines is the set $\{z : z_3 = 0, |z_1| = |z_2|\}$, and this is not a complex analytic set.

Example 2.4.6: Let us modify the example just a little bit to define $M \subset \mathbb{C}^3$ as

$$0 = 2\operatorname{Re}z_3 + |z_1|^2 + |z_1^2|^2 - |z_2|^2. \quad (2.29)$$

We have added a higher order term only, yet this modification produced a hypersurface containing no nontrivial complex analytic variety through the origin. By the theorem M can contain a variety if and only if there exists a 2×2 unitary $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_1^2 \end{bmatrix}. \quad (2.30)$$

Or in other words if $az_2 = z_1$ and $cz_2 = z_1^2$, where $|a|^2 + |c|^2 = 1$. Both a and c cannot be both zero. Hence either both z_1 and z_2 are zero or neither can be. We can solve to obtain $cz_1 = az_1^2$ and $cz_2 = a^2z_2^2$. From this we obtain $c = az_1$ and $c = a^2z_2$. Therefore for a fixed a and c we get at most two solutions, one $(0, 0)$ and one $(c/a, c/a^2)$. Thus there is no nontrivial variety through zero.

We can also give a wonderfully simple proof of the following simple version of the Diederich-Fornaess theorem (Theorem 2.6.5). The full version of Diederich-Fornaess allows singularities and does not require algebraicity but only real-analyticity.

Corollary 2.4.7 (Special case of Diederich-Fornaess). *Let $M \subset \mathbb{C}^n$ be a compact real-algebraic hypersurface in \mathbb{C}^n . Then at no point does M contain a germ of a nontrivial complex analytic variety.*

Proof. Let $r(z, \bar{z}) = 0$ be the defining equation for M and write $r(z, \bar{z}) = 2\operatorname{Re}h(z) + \|f(z)\|^2 - \|g(z)\|^2$, as in the theorem. Let $(X, p) \subset (M, p)$ be a germ of a complex variety. The theorem says that there exists a unitary U such that (X, p) is contained in some $V = V(h, f - Ug)$. The variety V is algebraic (defined by polynomials), so it has global set of defining equations (in particular it is a closed subset of \mathbb{C}^n). As $V \subset M$ and M is compact, then so is V . If V is compact, then find the point $q \in V$ such that $\|q\|^2$ is maximal for all points in V . Take a holomorphic map $\varphi: \mathbb{D} \rightarrow \mathbb{C}^n$ such that $\varphi(\mathbb{D}) \subset V$ and $\varphi(0) = q$. By noting that $\|\varphi(z)\|^2$ attains a maximum at 0, we use the maximum principle to show that φ is constant (exercise). Hence $V = V' \cup \{q\}$. If V' is nontrivial, apply same reasoning to V' to show that V must be a finite set of points. To see that the set is finite, remember that any variety V must be a locally finite union of manifolds. \square

2.5 Hermitian forms for real-analytic functions

Now suppose that $r(z, \bar{z})$ is Hermitian symmetric, but let us suppose that $z \mapsto r(z, \bar{z})$ is real-analytic rather than a polynomial. The first subtle point is about the domain of convergence. Convergence is not so simple for real analytic functions as it is for holomorphic functions. For example, if we have a holomorphic entire function $f: \mathbb{C}^n \rightarrow \mathbb{C}$, it has a power series around the origin that converges everywhere. On the other hand if we have a real-analytic function defined on all of \mathbb{C}^n (that is, it has a real power series near every point), it is no longer necessarily true that it will have a single everywhere convergent series. The problem is that while $r(z, \bar{z})$ might be analytic everywhere, but $r(z, \bar{w})$ may only be analytic in some neighborhood of $z = w$ and not in all of $\mathbb{C}^n \times \mathbb{C}^n$.

For every point p , we can however find a neighborhood U of p , such that $r(z, \bar{w})$ has a series that converges in $U \times U^*$ (remember U^* is the conjugate of U).

Now we come to the second subtle issue. When we write the coefficients of $r(z, \bar{z})$ as a matrix, it will be an infinite matrix. We can do a lot of the same computations formally as before, but we may have to then prove things about convergence. Furthermore, the mappings we get in the computations have infinite dimensional range, so we have to make sense of that. We will replace \mathbb{C}^m with a Hilbert space (we will assume it is separable). Remember a basis of a Hilbert space is not the algebraic basis, but the analytic Hilbert space one. If a countable set $\{e_j\}$ is an orthonormal basis of a separable Hilbert space \mathcal{H} , then every $p \in \mathcal{H}$ can be written as $p = \sum_{j=1}^{\infty} c_j e_j$. So you can simply think of ℓ^2 (the space of square summable sequences) for \mathcal{H} .

Definition 2.5.1. Let \mathcal{H} be a separable Hilbert space and let $\{e_j\}$ be an orthonormal basis. Then $f: \mathbb{C}^n \rightarrow \mathcal{H}$ is said to be holomorphic if and only if

$$\langle f(z), e_j \rangle \tag{2.31}$$

is holomorphic for every j .

Exercise 2.5.1: Let $U \subset \mathbb{C}^n$ be open and $f: U \rightarrow \mathcal{H}$ be holomorphic with respect to one basis. Show that it is holomorphic with respect to any basis.

Exercise 2.5.2: Let $U \subset \mathbb{C}^n$ be open and $f: U \rightarrow \mathcal{H}$ be holomorphic. Suppose that $0 \in U$. Show that you can write $f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}$, where $c_{\alpha} \in \mathcal{H}$.

If we take a Hermitian symmetric function $r(z, \bar{z})$ converging in a neighborhood $U \times U^*$ of the origin, then we can formally

$$r(z, \bar{z}) = \mathcal{Z}^* C \mathcal{Z}, \quad (2.32)$$

where C is the matrix of coefficients and \mathcal{Z} is an infinite vector of all the monomials. Now \mathcal{Z} does not map all of U into ℓ^2 and C may not even be a bounded operator. We have to be a little careful. We can always rescale the variables to ensure that the series for $r(z, \bar{z})$ has a domain of convergence that includes the point $z = (1, 1, \dots, 1)$ in its interior. Then r converges on the polydisc $\Delta = \Delta_1(0)$ and complexifies into $\Delta \times \Delta$. Furthermore \mathcal{Z} maps Δ into ℓ^2 as can be easily checked by noting the geometric series. As $(1, \dots, 1)$ is in the domain of convergence, we can plug in to \mathcal{Z} and note absolute convergence. The vector \mathcal{Z} becomes $(1, 1, 1, \dots)$ and thus

$$\sum_{\alpha\beta} |c_{\alpha\beta}| = \|C\|_1 < \infty. \quad (2.33)$$

Proposition 2.5.2. If $A = [a_{jk}]$ is an infinite matrix such that

$$\sum_{j,k=1}^{\infty} |a_{jk}|^2 < \infty, \quad (2.34)$$

then A defines a unique bounded operator on ℓ^2 with orthonormal basis $\{e_j\}$ such that $\langle Ae_j, e_k \rangle = a_{jk}$.

In fact, A is a compact (completely continuous) operator. We leave that as an exercise.

Proof. Write $M = \sum_{j,k=1}^{\infty} |a_{jk}|^2$. We have in particular that for each k

$$\sum_{j=1}^{\infty} |a_{jk}|^2 < \infty. \quad (2.35)$$

Thus define the operator A on the basis elements as

$$Ae_k = \sum_{j=1}^{\infty} a_{jk} e_j. \quad (2.36)$$

we can extend linearly to all finite combinations of basis elements. If u and v are finite combinations of basis elements it is not hard to see that we must get (the details are left to the reader)

$$|\langle Au, v \rangle| \leq \|Au\| \|v\| \leq \sqrt{M} \|u\| \|v\|. \quad (2.37)$$

By continuity this is true for all v . Letting $v = Au$ we find

$$\|Au\|^2 \leq \sqrt{M} \|u\| \|Au\|, \quad (2.38)$$

or in other words $\|Au\| \leq \sqrt{M} \|u\|$. Thus, A is bounded and we can extend to all of ℓ^2 .

We leave that A is unique to the reader. □

In fact, if a linear operator on ℓ^2 is bounded, then it is defined by a matrix, but we will not prove this fact although it is not hard with some basic functional analysis. See [AG93].

Thus if C is as in the discussion before the proposition, the proposition says that C defines a bounded operator on \mathcal{H} (note that C being absolutely summable implies it is square summable).

Proposition 2.5.3. *Let $U \subset \mathbb{C}^n$ be open. Suppose that $F: U \rightarrow \ell^2$ is holomorphic such that the series for $\|F(z)\|^2$ converges on a neighborhood of the closed unit polydisc $\bar{\Delta}$. Then the matrix of coefficients of $F(z)$ defines a bounded operator on ℓ^2 .*

Here by the matrix of coefficients of $F(z)$ we mean a matrix, which we also call F , such that $F\mathcal{Z} = F(z)$, where \mathcal{Z} is the infinite column vector of monomials in z .

Proof. Write $F(z) = \sum_{\alpha} f_{\alpha} z^{\alpha}$, where f_{α} is an infinite column vector. Consider the function $\|F(z)\|^2$. Formally we have that $\|F(z)\|^2 = \mathcal{Z}^* F^* F \mathcal{Z}$, and we know that this series converges as $F(z)$ is ℓ^2 valued. By the preceding discussion we have that $\|F^* F\|_1 < \infty$. Writing out what that means and only looking at the diagonal entries we have that

$$\sum_{\alpha} |\langle f_{\alpha}, f_{\alpha} \rangle| \leq \|F^* F\|_1 < \infty. \quad (2.39)$$

Therefore the matrix F satisfies the hypothesis of Proposition 2.5.2. \square

We prove the following theorem. To apply the theorem, you may have to do a change of coordinates. Notice the quantitative aspects of the theorem.

Theorem 2.5.4 (D'Angelo). *Let $r(z, \bar{z})$ be a Hermitian symmetric real-analytic function vanishing at p such that the series converges in a neighborhood of $\Delta \times \Delta^*$ for a polydisc Δ centered at p . Then there exist a holomorphic function $h: \Delta \rightarrow \mathbb{C}$ and holomorphic mappings $f: \Delta \rightarrow \ell^2$ and $g: \Delta \rightarrow \ell^2$, such that*

$$r(z, \bar{z}) = 2 \operatorname{Re} h(z) + \|f(z)\|^2 - \|g(z)\|^2 \quad (2.40)$$

and $h(p) = 0$, $f(p) = 0$, and $g(p) = 0$.

Alternatively, we can find two holomorphic mappings $F: \Delta \rightarrow \ell^2$ and $G: \Delta \rightarrow \ell^2$, such that

$$r(z, \bar{z}) = \|F(z)\|^2 - \|G(z)\|^2, \quad (2.41)$$

but we cannot guarantee that F or G vanish at p .

Proof. Without loss of generality we can assume that after an affine change of variables $p = 0$ and Δ is the unit polydisc.

It is not hard to find $h(z) = r(z, 0)$. Therefore, we will simply prove that we can write $r(z, \bar{z}) = \|F(z)\|^2 - \|G(z)\|^2$. The result will then easily follow.

We note that by the discussion above, the coefficient matrix C or $r(z, \bar{z})$ defines a bounded operator on ℓ^2 . We wish to decompose $C = A^* A - B^* B$. For example, write $A = \frac{1}{2}(C + I)$ and $B = \frac{1}{2}(C - I)$.

$$r(z, \bar{z}) = \langle C\mathcal{Z}, \mathcal{Z} \rangle = \langle A\mathcal{Z}, A\mathcal{Z} \rangle - \langle B\mathcal{Z}, B\mathcal{Z} \rangle. \quad (2.42)$$

It is not hard to check (exercise) that $A\mathcal{L}$ defines a holomorphic mapping from Δ to ℓ^2 . Same with $B\mathcal{L}$. \square

We will also require the theorem about unitary operators extended to the infinite dimensional setting.

Lemma 2.5.5 (D'Angelo). *Let $V \subset \mathbb{C}^n$ be a connected open neighborhood of p . Let $F: V \rightarrow \ell^2$ and $G: V \rightarrow \ell^2$ be holomorphic maps such that $\|F(z)\|^2 = \|G(z)\|^2$ for all $z \in V$. Write $F(z) = \sum f_\alpha(z-p)^\alpha$ and $G(z) = \sum g_\alpha(z-p)^\alpha$ (where f_α and g_α are in ℓ^2).*

Then there exists an isometry $U: \overline{\text{span}\{f_\alpha\}} \rightarrow \overline{\text{span}\{g_\alpha\}}$ such that $F(z) = UG(z)$.

Proof. As before translate and scale coordinates such that F is holomorphic in a neighborhood of the closed unit polydisc $\bar{\Delta}$ and let $p = 0$. Then the series of $\|F(z)\|^2$ complexifies to $\Delta \times \Delta^*$. Let C be the matrix of coefficients of the function $\|F(z)\|^2$, and as before C defines a bounded operator. Let F denote the matrix for $F(z)$ such that $F(z) = F\mathcal{L}$. Similarly with G . We therefore have that $C = F^*F = G^*G$. As C is bounded, so is F and G .

Now C is positive semidefinite and thus has a unique positive square root P by a standard theorem in functional analysis (for example [Rud91], Theorem 12.33). The root is determined up to partial isometry, that is, exactly the statement of the theorem above. Note that $R(F)$ is definitely contained in the closure of the range of F as an operator and same with G . \square

2.6 Complex varieties in real-analytic hypersurfaces

A *real-analytic hypersurface* is a real submanifold M of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ of real dimension $2n - 1$ that is defined by

$$M = \{z \in \mathbb{C}^n : r(z, \bar{z}) = 0\}$$

for some real-valued (Hermitian symmetric) real-analytic function r . For simplicity, we will assume that M is really a submanifold and is not singular.

Let $p \in M$ be a point and we decompose r as

$$r(z, \bar{z}) = 2\operatorname{Re}h(z) + \|f(z)\|^2 - \|g(z)\|^2, \quad (2.43)$$

where h , f and g vanish at p . Suppose that through p there is a positive dimensional complex variety contained in M . That is, there exists a nonconstant holomorphic function $\varphi: \mathbb{D} \rightarrow \mathbb{C}^n$ such that $\varphi(\mathbb{D}) \subset M$, $\varphi(0) = p$. That is, we have that for all $t \in \mathbb{D}$

$$r(\varphi(t), \overline{\varphi(t)}) = 0. \quad (2.44)$$

Just as before, we treat \bar{t} as a separate variable and set it to zero, to find that $r(\varphi(t), \bar{p}) = 0$. Hence, as before h vanishes on $\varphi(\mathbb{D})$, and hence on any germ of a complex variety in M through p .

Definition 2.6.1. Let U be a neighborhood of p such that $r(z, \bar{z})$ complexifies into $U \times U^*$. Then $\Sigma_p = \{z \in U : r(z, p) = 0\}$ defines the so-called *Segre variety* Σ_p for M at p

Do note that the definition does depend on U and r as before. As a warm up, and to illustrate the use of the Segre variety, let us prove a result about complex varieties of (complex) codimension 1 lying in a hypersurface. We will show that if such a variety goes through every point in M , then M has locally no biholomorphic invariants (M is Levi-flat, see below).

First let us define a *Levi-flat* hypersurface as a real codimension 1 submanifold $M \subset \mathbb{C}^n$ such that through every point $p \in M$, there exists a germ of a complex hypervariety (subvariety of dimension $n - 1$) (X, p) such that $(X, p) \subset (M, p)$. Being Levi-flat is a degenerate condition. An example of a Levi-flat hypersurface is the boundary of the polydisc, which is Levi-flat away from the ‘‘corners’’ where it is not a smooth submanifold.

Proposition 2.6.2. *Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface (nonsingular) If at p there is a germ of a complex subvariety (X, p) of (complex) dimension $n - 1$ such that $(X, p) \subset (M, p)$, then (X, p) is the unique nonsingular complex manifold through p and there exists a local defining function r for M such that $(X, p) = (\Sigma, p)$.*

Furthermore, if M is Levi-flat, then there exists a local biholomorphic change of coordinates near p that takes M into the manifold given by

$$\operatorname{Im} z_1 = 0. \quad (2.45)$$

Proof. First let $p = 0$. As M is a submanifold, we can find a defining function $r(z, \bar{z})$ for M near 0 that has linear terms (M is locally a graph over some real hyperplane after all). Fix a neighborhood U such that r complexifies to $U \times U^*$. We note that all the linear terms must be in $r(z, \bar{0})$. Thus Σ_0 is a submanifold. That means that there can be only a unique complex variety through 0 and that is Σ_0 (exercise).

If M is Levi-flat, the same is true for every point $p \in M$. Let $\varphi(t) \rightarrow M$ be a real-analytic function, then

$$r(z, \overline{\varphi(t)}) = 0. \quad (2.46)$$

We can pick φ in a way so that the image is not contained in Σ_p . Hence $r(z, \overline{\varphi(t)})$ is an infinite family of complex manifolds.

We treat \bar{t} as a complex variable in $t \mapsto r(z, \overline{\varphi(t)})$. As $r(z, \bar{p})$ is a graph near p , we can pick φ such that the complex derivative in \bar{t} of $t \mapsto r(z, \overline{\varphi(t)})$ is nonzero. Then by the holomorphic implicit function theorem we can write t as a function z . The function t is real-valued on M . We can thus perform the change of variables as given. The details are left to the reader. \square

Before we prove the analytic analogue of the theorem about germs of complex varieties in a real hypersurface we need a little bit of notation.

Definition 2.6.3. Let $r(z, \bar{z}) = 2\operatorname{Re} h(z) + \|f(z)\|^2 - \|g(z)\|^2$. Let p be a point such that h , f and g vanish at p . Furthermore suppose that h , f , and g are holomorphic in a polydisc Δ centered at p . Let

0 denote the map to ℓ^2 that is the constant element zero for all z . Define $\tilde{f} = f \oplus 0$ and $\tilde{g} = g \oplus 0$. Let U be a unitary operator on ℓ^2 . We define the family of ideals (in $\mathcal{O}(\Delta)$) as

$$I(U, p) \stackrel{\text{def}}{=} (h, \tilde{f} - U\tilde{g}). \quad (2.47)$$

That is, the ideals defined by the (infinitely many) equations $h(z) = 0$ and $\tilde{f}(z) = U\tilde{g}(z)$. Define

$$V(U, p) \stackrel{\text{def}}{=} \{z \in \Delta : \psi(z) = 0, \forall \psi \in I(U, p)\}. \quad (2.48)$$

A priori, $I(U, p)$ depends on the holomorphic decomposition and on the defining function. However, it can be shown that the ideals are independent if taken as ideals in \mathcal{O}_p . We will not show this fact, see [D'A93].

We prove the following theorem. Notice the quantitative aspects of the theorem.

Theorem 2.6.4 (D'Angelo). *Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface defined by $r(z, \bar{z}) = 0$ near p and let Δ be a polydisc centered at p such that the series for r converges in a neighborhood of $\overline{\Delta \times \Delta^*}$. We let*

$$r(z, \bar{z}) = 2\operatorname{Re}h(z) + \|f(z)\|^2 - \|g(z)\|^2, \quad (2.49)$$

be the holomorphic decomposition of r where h , f , and g are holomorphic in Δ and vanish at p , and f and g are ℓ^2 valued.

Then M contains a germ of complex variety at p if and only if there exists a unitary U such that $V(U, p)$ is nontrivial.

In fact, $V = V(U, p) \subset M$ for every unitary U , and if (X, p) is an irreducible germ of a complex variety such that $(X, p) \subset (M, p)$ then there exists a unitary U such that $(X, p) \subset (V, p)$. In particular, X lies in a subvariety of Δ .

Proof. The proof is very similar to the proof of theorem Theorem 2.4.3. We can assume that $p = 0$ and Δ is the unit polydisc.

Let us first start with the easy direction. Without loss of generality we can assume that $g = \tilde{g}$ and $f = \tilde{f}$ as in Definition 2.6.3. Suppose that there exists the unitary U such that $V(U, 0)$ is nontrivial. For a unitary U we have that $\|Ug(z)\|^2 = \|g(z)\|^2$. Thus if z is such that $f(z) = Ug(z)$ and $h(z) = 0$, then $2\operatorname{Re}h(z) + \|f(z)\|^2 - \|g(z)\|^2 = 0$. Thus $V(U, 0) \subset M$.

Now for the other direction, suppose that there is a germ of an irreducible complex variety through p in M . Take a representative X of this germ. As before, for any holomorphic mapping $\varphi: \mathbb{D} \rightarrow X \subset M$ with $\varphi(0) = p$, we have $h(\varphi(t)) = 0$ for all $t \in \mathbb{D}$. Thus for all $t \in \mathbb{D}$ we have $\|f(\varphi(t))\|^2 = \|g(\varphi(t))\|^2$. Similarly as before we parametrize a neighborhood of a nonsingular point of X using a map $\Phi: \Delta' \rightarrow X$, for a polydisc Δ' of same dimension as X such that for all $z \in \Delta'$

$$\|f(\Phi(z))\|^2 = \|g(\Phi(z))\|^2. \quad (2.50)$$

We have seen (Lemma 2.5.5) that this implies the existence of an isometry U on a subspace H_1 that is the closed span of the coefficients of $g(\Phi(z))$. The range is a subspace H_2 that is the closed

span of the coefficients of $f(\Phi(z))$. It is not necessarily true that this must extend to a unitary operator on ℓ^2 (for example it could happen that codimension of H_2 is finite, while codimension of H_1 is infinite).

But we are allowed to direct sum with a zero component, producing \tilde{f} and \tilde{g} , ensuring that the codimensions are infinite. Since all infinite dimensional separable Hilbert spaces are isometric, we can extend U to a unitary operator on the resulting space. The result follows. \square

Notice that the theorem implies that any germ of a subvariety extends to a subvariety of Δ . We can use this fact to prove the full strength of the Diederich-Fornaess theorem. Again the theorem also works if the hypersurface has singularities, but we will not worry about this.

Theorem 2.6.5 (Diederich-Fornaess '78 [DF78]). *Let $M \subset \mathbb{C}^n$ be a compact real-analytic hypersurface in \mathbb{C}^n . Then at no point does M contain a germ of a nontrivial complex analytic variety.*

Proof. Let S be the union of all (local) complex analytic varieties contained in M . As M is compact, then \bar{S} is compact, so let $p \in \bar{S}$ be the point furthest away from the origin. Then there exists an affine holomorphic function $\psi(z) = Lz + z_0$ that attains its maximum on \bar{S} exactly at p , for example suppose $\psi(p) = 1$. Furthermore, when $\delta > 0$ is small enough, then there is some $\varepsilon > 0$ such that such that the set $\{q \in \bar{S} : |\psi(q)| > 1 - \varepsilon\}$ lies within a ball of radius δ around p .

Now look at the defining equation r for M near p . There is some $\alpha > 0$ such that when we rescale, Theorem 2.6.4 applies at p and says that any complex variety X through p is contained in a subvariety of $\Delta_\alpha(p) \cap M$. In fact, Theorem 2.6.4 applies in a small neighborhood of p . That is, there is some $\beta > 0$ (smaller than α) such that for every $q \in B(\beta; p) \cap M$ (where $B(\beta; p)$ is the ball of radius β centered at p), every germ of a complex variety through q extends to $\Delta_\alpha(p)$ (the p is not a typo).

If we pick $q \in S$ close enough to p , for example, $\|p - q\| < \min\{\alpha, \beta, \delta\}$, then when we find the subvariety \tilde{X} of $\Delta_\alpha(p) \cap M$ that contains X and we notice that ψ attains its maximum on \tilde{X} away from the boundary of $\Delta_\alpha(p)$ (as $\tilde{X} \subset S$).

A holomorphic function cannot attain a maximum on a variety unless it is constant on the variety.

Exercise 2.6.1: *Prove maximum principle for varieties. If a holomorphic f attains a maximum on a variety \tilde{X} , then f restricted to \tilde{X} is constant. Hint: parametrize a curve in the variety near the point*

Thus the variety must lie within some $\{\psi = c\}$ for some constant c with $|c| < 1$. But δ could have been picked small enough such that $\{\psi = c\} \cap \bar{S}$ is a compact subset of $\Delta_\alpha(p)$ and so \tilde{X} is a compact subset of $\Delta_\alpha(p)$. By similar reasoning as before \tilde{X} has to be trivial. \square

Chapter 3

Proper Maps of Balls

3.1 Proper maps of balls

We have seen proper mappings before. They are continuous maps that pull back compact sets to compact sets. We have seen that they “take boundary to boundary.” Proper maps can be viewed as a “poor man’s biholomorphism,” as they possess one necessary (but not sufficient) condition for having a holomorphic inverse.

We have seen that there are no proper from the polydisc to the ball. Let us consider proper maps between balls then. Let us start with dimension 1.

Proposition 3.1.1. *Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be a proper map. Then*

$$f(z) = e^{i\theta} \prod_{j=1}^k \frac{z - a_j}{1 - \bar{a}_j z}, \quad (3.1)$$

for some real θ and some $a_j \in \mathbb{D}$ (that is, f is a finite Blaschke product).

Proof. Consider $E = f^{-1}(0)$. E is a subvariety of \mathbb{D} and it is compact by properness. Therefore, it is a finite set.

Exercise 3.1.1: *Finish the proof.*

□

How about in higher dimensions. Well the story is no longer so simple.

Theorem 3.1.2 (Pinčuk / Alexander ’77 (complicated history...)). *If $f: \mathbb{B}_n \rightarrow \mathbb{B}_n$ is a proper holomorphic map and $n \geq 2$, then f is an automorphism of \mathbb{B}_n .*

Remark 3.1.3. The theorem has natural generalizations to more complicated domains. For example, proper self-mappings of strongly pseudoconvex domains are automorphisms.

An automorphism of the ball is a linear fractional transformation (See Rudin [Rud80]). We will write $\text{Aut}(\mathbb{B}_n)$ for the group of automorphisms. The group happens to be equivalent to the real Lie group $SU(n, 1)$. And that is the way we will want to think of it, and we will give details in section 3.5. We can write it in the following explicit way

$$f(z) = U \frac{w - L_w z}{1 - \langle z, w \rangle} \quad (3.2)$$

for some $w \in \mathbb{B}_n$, a unitary matrix U , and a linear map

$$L_w z = sz + \frac{\langle z, w \rangle}{s+1} w, \quad \text{where } s = \sqrt{1 - \|w\|^2}, \quad (3.3)$$

although we will not find use for this form. It is interesting to note that the automorphisms do look like Blaschke factors in one dimension. We omit the proof of the theorem.

Notice that $z \mapsto z^d$ is a proper map of the disc to the disc for any $d \in \mathbb{N}$. What happens to z^d if $n \geq 2$? Well, it is still there, we just need to look at tensor products and we need to allow the target dimension to become larger.

So what if the target and domain are not of the same dimension? First let us start with the easiest case.

Proposition 3.1.4. *Let $f: \mathbb{B}_n \rightarrow \mathbb{B}_N$ be a proper holomorphic map. Then $N \geq n$.*

Proof. A proper map pulls back compact sets to compact sets. The inverse image $f^{-1}(0)$ is the simultaneous vanishing of the components of f . As there are N components of f , the dimension of the variety $f^{-1}(0)$ is at least $n - N$. Since the only compact variety is of dimension 0, we see that $n - N \leq 0$ or in other words $n \leq N$. \square

As we have foreshadowed, there do exist nontrivial mappings when the target dimension is larger than n . As long as the maps are nice up to the boundary, there is some bound on their complexity.

Theorem 3.1.5 (Forstnerič '89 [For89]). *Let $N \geq n \geq 2$ be given. If $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ is a proper holomorphic mapping that is smooth (C^∞) up to the boundary, then F is rational.*

In fact, the degree of F is bounded by some number $D(n, N)$.

The degree of a rational mapping is the maximum of the degree of the numerator and the denominator when written in lowest terms.

Note that the requirement that the mapping extends to the boundary is necessary if $N > n$. There exist non-rational maps from \mathbb{B}_n to \mathbb{B}_N that are only continuous up to the boundary. In fact such maps already exist when $N = n + 1$, see [Dor90].

We will want to classify rational proper mappings of balls. We will therefore need some notion of equivalence.

Definition 3.1.6. We say $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ and $G: \mathbb{B}_n \rightarrow \mathbb{B}_N$ are *spherically equivalent* if there exist $\chi \in \text{Aut}(\mathbb{B}_n)$ and $\tau \in \text{Aut}(\mathbb{B}_N)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}_n & \xrightarrow{F} & \mathbb{B}_N \\ \downarrow \chi & & \downarrow \tau \\ \mathbb{B}_n & \xrightarrow{G} & \mathbb{B}_N \end{array} \quad (3.4)$$

To simplify notation, we also define spherical equivalence for different target dimensions. Suppose that $N < M$ and $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ and $G: \mathbb{B}_n \rightarrow \mathbb{B}_M$. Let $F \oplus 0$ denote the mapping F with $M - N$ zero components appended. We say F and G are spherically equivalent if $F \oplus 0$ and G are spherically equivalent.

As an example, we get the following complete classification in the case of low dimensions.

Theorem 3.1.7 (Faran '82 [Far82]). *If $F: \mathbb{B}_2 \rightarrow \mathbb{B}_3$ is a proper mapping that is C^3 up to the boundary, then F is spherically equivalent to one of*

$$\begin{array}{ll} (z_1, z_2) \mapsto (z_1, z_2, 0) & \text{("Linear embedding")} \\ (z_1, z_2) \mapsto (z_1, z_1 z_2, z_2^2) & \text{("Whitney map")} \\ (z_1, z_2) \mapsto (z_1^2, \sqrt{2} z_1 z_2, z_2^2) & \text{("Homogeneous")} \\ (z_1, z_2) \mapsto (z_1^3, \sqrt{3} z_1 z_2, z_2^3) & \text{("Faran map")} \end{array}$$

The proof is a complicated argument using moving frames. The theorem implies that the maximum possible degree for a map from \mathbb{B}_2 to \mathbb{B}_3 is 3.

Note that every mapping that Faran found is *monomial*. That is, every component is a single monomial. For higher n , some information is known. The following theorem is a result that is a culmination of successive improvements of theorems of Webster '79 [Web79], Faran '86 [Far86], Cima-Suffridge '83 [CS83], Huang '99 [Hua99], Huang-Ji '01 [HJ01], Hamada '05 [Ham05], and Huang-Ji-Xu '05 and '06 [HJX06]. Many of the later developments reduce the regularity to the boundary necessary for rationality. For simplicity we will state the theorem for rational maps.

Theorem 3.1.8. *Let $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ be rational proper and holomorphic.*

- (i) *If $n > 1$ and $N < 2n - 1$, then F is spherically equivalent to the identity.*
- (ii) *If $n \geq 3$ and $N \leq 2n - 1$, then F is spherically equivalent to a monomial map of degree two or less.*
- (iii) *If $n \geq 4$ and $N \leq 3n - 4$, then F is spherically equivalent to a monomial map of degree two or less.*

As we will learn how to classify monomial mappings, and in fact all mappings of degree 2 or less, we do not give the formulas here.

D'Angelo conjectured that the sharp bound (the constant $D(n, N)$ from Forstnerič's theorem) for the degree of a rational proper mapping $F: \mathbb{B}_n \rightarrow \mathbb{B}_n$ is

$$\deg F \leq \begin{cases} 2N - 3 & \text{if } n = 2, \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases} \quad (3.5)$$

Obviously there is no degree bound when $n = 1$. As there are examples where equality is satisfied the inequalities are sharp if true. We will see how to construct the sharp maps when $n \geq 3$ in the next section.

3.2 Tensors and polynomial proper maps

What happened to z^d ? Let z be the identity map and consider

$$z^{\otimes d} = z \otimes z \otimes \cdots \otimes z. \quad (3.6)$$

In simple terms if $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_m)$ are mappings then, $f \otimes g$ means

$$f \otimes g = \bigoplus_{j=1}^n \bigoplus_{k=1}^m f_j g_k \quad (3.7)$$

For example, if $n = 2$ and $d = 2$, then $z^{\otimes 2}$ is the mapping

$$(z_1, z_2) \mapsto (z_1^2, z_1 z_2, z_2 z_1, z_2^2). \quad (3.8)$$

To see that $z^{\otimes d}$ is a proper mapping of balls, we simply check that it takes the unit sphere to the unit sphere. To see this we notice that

$$\|f \otimes g\| = \|f\| \|g\|. \quad (3.9)$$

Therefore

$$\|z^{\otimes d}\| = \|z\|^d. \quad (3.10)$$

Therefore if $\|z\| = 1$, then $\|z^{\otimes d}\| = 1$.

We notice that some of the components on $z^{\otimes d}$ are redundant. For example take the example with $n = 2$ and $d = 2$. A priori, $z^{\otimes 2}$ takes \mathbb{B}_2 to \mathbb{B}_4 . We can however compose with a unitary mapping to obtain

$$(z_1, z_2) \mapsto (z_1^2, \sqrt{2} z_1 z_2, z_2^2, 0). \quad (3.11)$$

We can ignore the trailing zero and define

$$H_{2,2}(z_1, z_2) \stackrel{\text{def}}{=} (z_1^2, \sqrt{2} z_1 z_2, z_2^2). \quad (3.12)$$

So $H_{2,2}$ is a proper map from \mathbb{B}_2 to \mathbb{B}_3 .

We define $H_{n,d}$ in a similar fashion.

A mapping is a homogeneous polynomial of degree d if $F(tz) = t^d F(z)$.

Theorem 3.2.1 (Rudin '84). *If $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ is proper and a homogeneous polynomial of degree d , then F is spherically equivalent (actually unitarily equivalent) to $H_{n,d}$.*

There is an elegant proof of this theorem by D'Angelo.

Proof. On the sphere we have that

$$\|F(z)\| = \|z\|^d = \|z^{\oplus d}\| = \|H_{n,d}(z)\| = 1 \quad (3.13)$$

Now take any $w \in \mathbb{C}^n$. Write $w = tz$, where $t \geq 0$ and $\|z\| = 1$. By homogeneity we have

$$\|F(tz)\| = \|t^d F(z)\| = t^d \|F(z)\| = t^d \|H_{n,d}(z)\| = \|t^d H_{n,d}(z)\| = \|H_{n,d}(tz)\| \quad (3.14)$$

Hence $\|F(w)\| = \|H_{n,d}(w)\|$ for all $w \in \mathbb{C}^n$. As all components of $H_{n,d}$ are linearly independent, we know that we may only need to add zero components to $H_{n,d}$ and obtain a constant unitary matrix such that

$$F = U(H_{n,d} \oplus 0) \quad (3.15)$$

If F already had the same number of components as $H_{n,d}$ then the zeros are not necessary. \square

We can also only tensor on a subspace. For example, let $z = (z', z_n)$. Then we can create the so-called *Whitney map*

$$z' \oplus (z_n \otimes z) = (z_1, z_2, \dots, z_{n-1}, z_n z_1, z_n z_2, \dots, z_n^2). \quad (3.16)$$

Let us see that this map takes sphere to sphere. Suppose $\|z\| = 1$. Then

$$\|z' \oplus (z_n \otimes z)\|^2 = \|z'\|^2 + \|z_n \otimes z\|^2 = \|z'\|^2 + |z_n|^2 \|z\|^2 = \|z'\|^2 + |z_n|^2 = \|z\|^2 = 1. \quad (3.17)$$

The map has $2n - 1$ components and is of degree 2. Therefore, for $n \geq 3$, this map gives equality in the conjectured degree bound.

To create other maps with high degree and low number of terms we could simply repeat the procedure. For example to create a degree 3 map we write

$$z' \oplus (z_n \otimes (z' \oplus (z_n \otimes z))). \quad (3.18)$$

The procedure is simply to isolate the z_n^2 component from the Whitney map and tensor on that 1 dimensional subspace with the identity again. This map and maps constructed in the same way are sometimes called the *generalized Whitney maps*.

In general we can tensor by the identity on any subspace to obtain another proper map of balls. For example take a proper map of balls written as $F(z) = (f_1(z), f_2(z))$ for two vector valued maps f_1 and f_2 . Then we can simply obtain the map

$$f_1 \oplus (f_2 \otimes z) = (f_1(z), f_2(z)z_1, \dots, f_2(z)z_n). \quad (3.19)$$

The proof that this new map takes the sphere to the sphere is virtually identical to the computation above. We notice that when working on the level of the norm squared, tensoring is simply the more convenient multiplication of polynomials.

3.3 Monomial proper maps

Before tackling rational proper maps, let us first say a little bit about the monomial maps (every component is a single monomial). Further, let us classify all the degree 2 monomial proper ball maps. We will use this classification later.

We start with setting up a real-algebraic problem. Let $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ be a proper monomial map. That is,

$$\|F(z)\|^2 = 1 \quad \text{when} \quad \|z\|^2 = 1. \quad (3.20)$$

If a component of F is z^α , then the corresponding term in $\|F(z)\|^2$ is $|cz^\alpha|^2$. Therefore, if we replace $|z_1|^2$ with x_1 , $|z_2|^2$ with x_2 , etc. . . , then $|cz^\alpha|^2$ becomes $\sqrt{|c|}x^\alpha$. The form $\|z\|^2$ becomes $x_1 + \cdots + x_n$. Therefore, assuming that the components of F were linearly independent (were distinct monomials) we obtain a real polynomial $p(x)$ with positive coefficients such that

$$p(x) = 1 \quad \text{when} \quad \sum_j x_j = 1, \quad (3.21)$$

and such that p has N terms. It is easy to see that up to multiplying each component by some $e^{i\theta}$ we can start with $p(x)$ and obtain a proper map of balls.

Lemma 3.3.1. *If $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ is a proper monomial map, then F is spherically equivalent to a monomial map taking the origin to the origin by post-composing with a diagonal matrix.*

Proof. Take F and produce $p(x)$ as above. If $F(0) \neq 0$, then $p(0) \neq 0$. In other words, $p(x)$ has a constant term $p(0)$. We note that $0 < p(0) < 1$. If $p(0) = 1$, then $p(x) = 1$ and F was not a proper map of balls. It is easy to see that $p(0) > 1$ is impossible.

Then $\frac{p(x)-p(0)}{1-p(0)}$ is a polynomial with nonnegative coefficients with no constant term that is one on $x_1 + \cdots + x_n = 1$. The equivalence of the induced maps is by a diagonal matrix. \square

Lemma 3.3.2. *If $F: \mathbb{B}_n \rightarrow \mathbb{B}_N$ is a proper monomial map of degree 2 or less, then F is spherically equivalent to one map of the form*

$$Dz \oplus ((\sqrt{I - D^2} z) \otimes z), \quad (3.22)$$

where $D = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix}$ with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1$. If $F(0) = 0$, then equivalence is by permuting the variables and components of the map and post-composing with unitary matrix.

Proof. Take F and construct $p(x)$ as above. We obtain a polynomial $p(x)$ of degree 2, such that $p - 1$ is divisible by $(x_1 + \cdots + x_n - 1)$, $p(0) = 0$, and all coefficients of p are nonnegative.

Write $p = p_1 + p_2$ where p_1 is of degree 1, and p_2 is of degree 2. Homogenize $p - 1$ by $p_1(x)t + p_2(x) - t^2$ and note that this polynomial is divisible by $(x_1 + \cdots + x_n - t)$. By setting $t = 0$ we note that p_2 is divisible by $(x_1 + \cdots + x_n)$. Write a polynomial $q = p_1 + \frac{p_2}{x_1 + \cdots + x_n}$. The

polynomial q is of degree 1, $q(0) = 0$, and $q - 1$ is also divisible by $(x_1 + \dots + x_n - 1)$. Hence, $q = x_1 + \dots + x_n$.

So p was constructed from q by a partial tensoring operation, which in this language is just multiplication of a part of q by $(x_1 + \dots + x_n)$. The p_1 corresponds to Dz . Then obviously $\frac{p_2}{x_1 + \dots + x_n}$ must correspond to $\sqrt{I - D^2}z$, where $I - D^2 \geq 0$. By a permutation matrix, we can sort the entries in D . The unitary matrix comes about because of the translation from the real-algebraic setting back to proper mappings, possibly to account for components of the map not being linearly independent. Do note that the equivalence is taking into account possibly adding zero components. \square

We can symmetrize the tensor by a unitary matrix just as we did for the homogeneous map. For example, for $n = 2$ we have the following family of maps.

$$(z_1, z_2) \mapsto (\sqrt{s}z_1, \sqrt{t}z_2, \sqrt{1-s}z_1^2, \sqrt{2-s-t}z_1z_2, \sqrt{1-t}z_2^2), \quad 0 \leq s \leq t \leq 1. \quad (3.23)$$

Here we took $D = \begin{bmatrix} \sqrt{s} & 0 \\ 0 & \sqrt{t} \end{bmatrix}$ and then symmetrized the two components corresponding to z_1z_2 . We will see later that monomial maps taking the origin to the origin are spherically equivalent if and only if they are equivalent by a permutation of the variables and components and post-composing with a unitary. In particular, the maps above are inequivalent for different parameters (s, t) . It is not too difficult to prove this by directly applying the automorphisms and noting that distinct monomials are linearly independent. However, for degree 2 maps, the result will be automatic from classification of the underlying Hermitian forms.

Once we have that the monomial maps above are mutually inequivalent, then we see that for example for $n = 2$ the moduli space of maps of degree 2 or less is a closed triangle with one corner being the identity, another corner being the Whitney map and the last corner being $H_{2,2}$. See Figure 3.1.

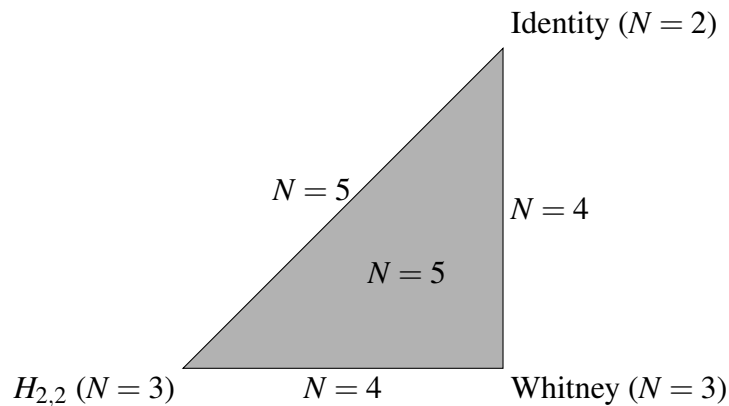


Figure 3.1: Moduli space of degree 2 proper maps of \mathbb{B}_2 to \mathbb{B}_N in the s, t plane. The minimal target dimension N is marked for each vertex, edge, and the interior.

Similarly, the moduli space for n dimensions is an $n + 1$ dimensional simplex with one vertex for the identity, and the other vertices for maps obtained by tensoring with the identity on $1, 2, \dots, n$ dimensional subspace. We will prove that this is the moduli space for all rational degree 2 proper maps of balls.

3.4 Projective setting and hyperquadrics

It turns out that to handle rational maps, it is easiest to work in the complex projective space. Rational maps become polynomials and linear fractional maps (automorphisms of the ball) become linear maps. We will be able to simply apply linear algebra.

Definition 3.4.1. The n -dimensional (*complex*) *projective space* $\mathbb{C}\mathbb{P}^n$, or just \mathbb{P}^n , is the space of complex lines through the origin in \mathbb{C}^{n+1} .

The space \mathbb{P}^n is an n dimensional complex manifold. If we look at all the points $z \in \mathbb{C}^{n+1}$ such that $z_{n+1} = 1$, then we get a one-to-one correspondence of \mathbb{C}^n with an open subset of \mathbb{P}^n . These coordinates are the so-called *inhomogeneous coordinates* on \mathbb{P}^n . We define a natural projection

$$\tau: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n \quad (3.24)$$

by taking a point $z \in \mathbb{C}^{n+1} \setminus \{0\}$ to the point in \mathbb{P}^n representing the unique line through z and 0 in \mathbb{C}^{n+1} . We will simply write τ regardless of n .

Exercise 3.4.1: Check that τ is holomorphic (check this in some set of inhomogeneous coordinates).

The “coordinates” of \mathbb{C}^{n+1} are called the *homogeneous coordinates*. Do note however that these are not actual coordinates as z_j is not an actual function on \mathbb{P}^n . The reason for doing this is the following proposition. First let us define homogeneity.

Definition 3.4.2. A polynomial $p(z)$, $z \in \mathbb{C}^{n+1}$ is said to be *homogeneous* of degree d if

$$p(tz) = t^d p(z) \quad (3.25)$$

for all $t \in \mathbb{C}$. A polynomial $p(z, \bar{z})$ is said to be *bihomogeneous* of *bidegree* (d, d) if

$$p(tz, \bar{z}) = p(z, t\bar{z}) = t^d p(z, \bar{z}) \quad (3.26)$$

for all $t \in \mathbb{C}$.

Proposition 3.4.3. A zero set of a homogeneous holomorphic polynomial $p(z)$ ($z \in \mathbb{C}^{n+1}$) is a well defined subset of \mathbb{P}^n . A zero set of a bihomogeneous polynomial $p(z, \bar{z})$ is a well defined subset.

Proof. Trivial: If $p(z) = 0$, then $p(tz) = 0$ for all t and therefore p is zero on the whole line through the origin. As \mathbb{P}^n is the space of lines, $p(z) = 0$ is a well defined set. Similarly with a bihomogeneous polynomial. \square

In fact by setting a variable to 1, we get a polynomial equation in n variables. Thus the zero sets above are actual varieties (real or complex). So while the homogeneous and bihomogeneous polynomials have no well defined values on \mathbb{P}^n , they do have well defined zero sets.

Let us now define hyperquadrics (including the sphere) in inhomogeneous coordinates. The hyperquadric $Q(a, b)$ is defined as

$$Q(a, b) \stackrel{\text{def}}{=} \{z \in \mathbb{C}^{a+b} : |z_1|^2 + \cdots + |z_a|^2 - |z_{a+1}|^2 - \cdots - |z_{a+b}|^2 = 1\}. \quad (3.27)$$

To extend $Q(a, b)$ to a subset of \mathbb{P}^n we define

$$HQ(A, B) \stackrel{\text{def}}{=} \{z \in \mathbb{C}^{A+B} : |z_1|^2 + \cdots + |z_A|^2 - |z_{A+1}|^2 - \cdots - |z_{A+B}|^2 = 0\}. \quad (3.28)$$

$HQ(A, B)$ is defined as a cone in \mathbb{C}^{A+B} and therefore we can identify it with a subset of \mathbb{P}^{A+B-1} . Therefore we will not make a distinction and write $HQ(A, B)$ both for the cone in \mathbb{C}^{A+B} and for the subset of \mathbb{P}^{A+B-1} . Now note that when restricted to the patch defined by $z_{A+B} = 1$, we get that $HQ(A, B) = Q(A, B-1)$.

Now suppose that we have a homogeneous (holomorphic) polynomial map

$$f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}. \quad (3.29)$$

Exercise 3.4.2: Show that a homogeneous f takes lines through the origin in \mathbb{C}^{n+1} to lines through the origin in \mathbb{C}^{m+1} .

Proposition 3.4.4. Let $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}$ be a homogeneous holomorphic polynomial map. Then $V = V(f) \subset \mathbb{P}^n$ be the common zero locus of the components of f . Then f induces a holomorphic map $F: \mathbb{P}^n \setminus V \rightarrow \mathbb{P}^m$ given by $F = \tau \circ f \circ \tau^{-1}$. On any pair of inhomogeneous coordinate patches, the map is rational.

We will simply call f a rational map. Generally we will divide through by any common factor. Without proof we note that this will mean that V will be of codimension 2 or more in \mathbb{P}^n .

Proof. The map is well defined by the exercise. Restrict to some inhomogeneous coordinate patch. If we are not in V , then one of the components of f is nonzero near some point, suppose that it is f_{m+1} . Now take the map

$$\tilde{f} = \left(\frac{f_1}{f_{m+1}}, \frac{f_2}{f_{m+1}}, \dots, \frac{f_m}{f_{m+1}} \right). \quad (3.30)$$

As f_{m+1}/f_{m+1} is always one. Thus \tilde{f} is a well defined rational map into the inhomogeneous coordinate patch where the $(m+1)$ th component of \mathbb{C}^{m+1} is one. \square

We state the next observation as a proposition. The proof is obvious.

Proposition 3.4.5. *Let $X \subset \mathbb{P}^n$ be a set such that $r(z, \bar{z})$ is a bihomogeneous polynomial of bidegree (d, d) that vanishes on X . Suppose that the signature pair of r is (a, b) . Decompose $r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2$ for f and g that are homogeneous of degree d . Then (f, g) induces a rational map of \mathbb{P}^n to \mathbb{P}^{a+b-1} . As long as X is not contained in the common zero set of f and g , then (f, g) is a rational map that takes an open subset of X to $HQ(a, b)$.*

In particular we are interested in $X = HQ(n, 1)$, which is the sphere S^{2n-1} in homogeneous coordinates. If $r(z, \bar{z})$ vanishes on $HQ(n, 1)$, then r is divisible by the defining equation of $HQ(n, 1)$. That is

$$r(z, \bar{z}) = q(z, \bar{z}) \left(|z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2 \right) \quad (3.31)$$

for some bihomogeneous $q(z, \bar{z})$. If the signature pair of $r(z, \bar{z})$ is $(N, 1)$, then when decomposed as

$$r(z, \bar{z}) = \|f(z)\|^2 - |g(z)|^2 \quad (3.32)$$

the map (f, g) takes (an open subset) $HQ(n, 1)$ to $HQ(N, 1)$. To make this claim we note that $HQ(n, 1)$ is a real hypersurface and thus cannot possibly be contained in the common zero set of f and g . Thus if we take $\frac{f}{g}$ and set $z_{n+1} = 1$, we obtain a rational mapping that takes sphere to sphere.

On the other hand, suppose that we have a rational mapping $\frac{f}{g}$ that takes $S^{2n-1} \subset \mathbb{C}^n$ to $S^{2N-1} \subset \mathbb{C}^N$. We homogenize f and g (call the homogenized versions \tilde{f} and \tilde{g}). Then the map (\tilde{f}, \tilde{g}) takes $HQ(n, 1)$ to $HQ(N, 1)$. If we suppose that f and g had no common factors and were linearly independent, then when we write

$$r(z, \bar{z}) = \|\tilde{f}(z)\|^2 - |\tilde{g}(z)|^2, \quad (3.33)$$

we note that r must have signature pair $(N, 1)$.

Thus we have an identification of rational maps of spheres with bihomogeneous polynomials vanishing on $HQ(n, 1)$ with signature pair $(N, 1)$. The trouble is that the $r(z, \bar{z})$ that we obtain has many different decompositions. We need to know how are the maps corresponding to two different decompositions related.

3.5 Hermitian forms

As before, $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product. Let A be a Hermitian matrix, then a Hermitian form is simply

$$\langle Az, z \rangle, \quad (3.34)$$

where $z \in \mathbb{C}^{n+1}$. As we will often talk of the zero sets of Hermitian forms, define

$$\mathbb{V}_A \stackrel{\text{def}}{=} \{z \in \mathbb{C}^{n+1} : \langle Az, z \rangle = 0\}. \quad (3.35)$$

Let $\mathcal{L} = \mathcal{L}_d$ be the degree d Veronese map

$$(z_1, \dots, z_{n+1}) \xrightarrow{\mathcal{L}} (z_1^d, z_1^{d-1}z_2, \dots, z_{n+1}^d), \quad (3.36)$$

that is, the map whose components are all the degree d monomials. As we saw before we will think of a bihomogeneous polynomial $r(z, \bar{z})$ as

$$r(z, \bar{z}) = \langle A \mathcal{L}, \mathcal{L} \rangle, \quad (3.37)$$

where A matrix of coefficients.

Let f be a rational map of \mathbb{P}^n to \mathbb{P}^N . In homogeneous coordinates, f is given by $N+1$ homogeneous polynomials $f = f_1, f_2, \dots, f_{N+1}$. Let us formulate what it means for f to take $\mathbb{V}_J \subset \mathbb{P}^n$ to $\mathbb{V}_V \subset \mathbb{P}^N$ for two Hermitian matrices J and V . We can simply plug f into the equation for \mathbb{V}_V to obtain

$$\langle Vf(z), f(z) \rangle = 0. \quad (3.38)$$

Then f takes \mathbb{V}_J to \mathbb{V}_V if and only if there exists a bihomogeneous real polynomial q such that

$$\langle Vf(z), f(z) \rangle = q(z, \bar{z}) \langle Jz, z \rangle. \quad (3.39)$$

To classify f that take \mathbb{V}_J to \mathbb{V}_V we first need to find out when do such mappings differ by an automorphism preserving \mathbb{V}_V .

Lemma 3.5.1. *Let V be a nonsingular Hermitian matrix. Let f and g be homogeneous polynomial maps of \mathbb{C}^{n+1} to \mathbb{C}^{N+1} such that*

$$\langle Vf(z), f(z) \rangle = \langle Vg(z), g(z) \rangle. \quad (3.40)$$

Suppose that f has linearly independent components. Then

$$g(z) = Cf(z) \quad (3.41)$$

*for some invertible matrix C such that $C^*VC = V$.*

Proof. We write $f(z) = F\mathcal{L}$ for some complex matrix F with linearly independent rows. Then

$$\langle Vf(z), f(z) \rangle = \langle VF\mathcal{L}, F\mathcal{L} \rangle = \langle F^*VF\mathcal{L}, \mathcal{L} \rangle. \quad (3.42)$$

Similarly, $g(z) = G\mathcal{L}$. Thus $\langle Vf(z), f(z) \rangle = \langle Vg(z), g(z) \rangle$ implies that

$$\langle (F^*VF - G^*VG)\mathcal{L}, \mathcal{L} \rangle = 0. \quad (3.43)$$

We have a real polynomial that is identically zero and so its coefficients are zero. Thus $F^*VF = G^*VG$. By permuting the monomials, we could suppose that $F = [F_1 \ F_2]$, where F_1 is an invertible matrix. We write $G = [G_1 \ G_2]$, where G_1 is square. Now $F^*VF = G^*VG$ implies $F_1^*VF_1 = G_1^*VG_1$. Let $C = G_1F_1^{-1}$. Clearly $C^*VC = V$ and $G_1 = CF_1$. We have our matrix C we simply need to show that $G_2 = CF_2$. $F^*VF = G^*VG$ also implies $F_1^*VF_2 = G_1^*VG_2$. Now compute

$$F_1^*VF_2 = G_1^*VG_2F_1^*C^*VG_2 = F_1^*VC^{-1}G_2.$$

As F_1 and V are invertible we get $G_2 = CF_2$ and thus $g(z) = Cf(z)$. \square

Lemma 3.5.1 says that if $\langle Vf(z), f(z) \rangle = \langle Vg(z), g(z) \rangle$, then f and g are equivalent up to a linear map of the target space preserving the form defined by V . Note that these linear maps (up to a scalar multiple) correspond exactly to linear fractional transformations that take \mathbb{V}_V to itself and preserve the sides of \mathbb{V}_V . As we are working in homogeneous coordinates, we must also always consider the possibility $\langle Vf(z), f(z) \rangle = \lambda \langle Vg(z), g(z) \rangle$ for $\lambda > 0$. We can then simply scale f or g and use the proposition. Note that we will only consider V where the number of positive and negative eigenvalues are not equal. Otherwise we would also want to consider maps that take V to $-V$. The V we will consider defines the ball and always has only one negative eigenvalue.

For the hyperquadric $HQ(a, b)$ we let V be the matrix with a ones and b negative ones on the diagonal. Thus for a sphere, V will always have N ones and one -1 . Similarly for J . Let $U(N, 1)$ be the set of matrices M such that $M^*VM = V$. $U(N, 1)$ is called the *generalized unitary group*. It is left as an easy exercise that up to scaling $U(N, 1)$ represent linear fractional transformations preserving the sphere. The scaling comes about because we are working in homogeneous coordinates. Therefore, the set of linear fractional transformations that preserve the sphere is the Lie group $SU(N, 1)$ (the *generalized special unitary group*), that is, matrices in $U(N, 1)$ with determinant 1.

Exercise 3.5.1: *Linear fractional transformations preserving the S^{2N-1} are in one-to-one correspondence with matrices in $SU(N, 1)$.*

Corollary 3.5.2. *Let f and g be rational CR maps of S^{2n-1} to S^{2N-1} and let V be a Hermitian form defining $HQ(N, 1)$. Let f_h and g_h be the corresponding homogeneous polynomial maps of \mathbb{C}^{n+1} to \mathbb{C}^{N+1} . Suppose that the components of f_h are linearly independent. Then f and g are spherically equivalent if and only if there exists a $T \in U(n, 1)$ and $\lambda > 0$, such that*

$$\langle Vf_h(Tz), f_h(Tz) \rangle = \lambda \langle Vg_h(z), g_h(z) \rangle \quad \text{for all } z \in \mathbb{C}^{n+1}. \quad (3.44)$$

The requirement that components of f_h are linearly independent simply translates to the requirement that the image of f does not lie in a hyperplane.

The proof follows immediately by putting together the results of the preceding discussion. Similar lemma follows for all hyperquadrics. The only technical issue that comes up with general hyperquadrics is when the number of positive and negative eigenvalues are equal, in which case we would have to allow λ to be negative.

Consequently, by working in projective space and with Hermitian forms rather than the maps themselves, we reduce the spherical equivalence problem to an equivalence problem using only the group of automorphisms of the source. Any real polynomial corresponds to a rational map $f: \mathbb{P}^n \rightarrow \mathbb{P}^N$ taking the zero set of the polynomial to a hyperquadric with the same signature as the coefficient matrix. Therefore to classify all maps taking one hyperquadric to another hyperquadric up to the group of linear fractional maps that fix the hyperquadric, we need only classify real polynomials that vanish on the source hyperquadric and have the correct signature.

Let J define $HQ(n, 1)$. If we have a Hermitian matrix B such that $\langle B\mathcal{L}_d, \mathcal{L}_d \rangle$ is zero on \mathbb{V}_J , then

$$\langle B\mathcal{L}_d, \mathcal{L}_d \rangle = \langle A\mathcal{L}_{d-1}, \mathcal{L}_{d-1} \rangle \langle Jz, z \rangle, \quad (3.45)$$

for some Hermitian matrix A . We take an automorphism preserving the set defined by J , that is, a matrix T such that $T^*JT = J$. To find all maps equivalent by an automorphism of the source we compute the canonical form of

$$\langle A\mathcal{L}_{d-1}(Tz), \mathcal{L}_{d-1}(Tz) \rangle \langle Jz, z \rangle = \langle \hat{T}^*A\hat{T}\mathcal{L}_{d-1}, \mathcal{L}_{d-1} \rangle \langle Jz, z \rangle, \quad (3.46)$$

where \hat{T} is the matrix defined by $\mathcal{L}_{d-1}(Tz) = \hat{T}\mathcal{L}_{d-1}(z)$. Hence, we will look for a canonical form of the pair (J, A) under $*$ -conjugation by \hat{T} and T . When $d = 2$, then $\hat{T} = T$ and matters become simpler.

3.6 Degree 2 maps

Let us consider degree-two CR maps of spheres. We will prove the following theorem.

Theorem 3.6.1. *Let $f: S^{2n-1} \rightarrow S^{2N-1}$, $n \geq 2$, be a rational CR map of degree 2. Then f is spherically equivalent to a monomial map.*

Proof. By the preceding section, to study degree 2 monomial sphere maps, we simply study the equation

$$\langle B\mathcal{L}, \mathcal{L} \rangle = \langle Az, z \rangle \langle Jz, z \rangle, \quad (3.47)$$

where \mathcal{L} is the degree two Veronese mapping, B is the matrix F^*VF , F is the matrix of coefficients of the map under consideration, V defines $HQ(N, 1)$, and J defines $HQ(n, 1)$.

If we start on the right hand side, we multiply $\langle Az, z \rangle \langle Jz, z \rangle$ and obtain a B , we simply need to check that B has the right signature pair, that is only one negative eigenvalue. By going through all possible Hermitian matrices A we get all degree-two maps of \mathbb{V}_J to all hyperquadrics. Then we restrict to those matrices A where the resulting matrix B has the right signature.

By composing with a unitary and removing zero components, we can assume that the components of f are linearly independent.

Exercise 3.6.1: *a) Show that if f is a CR sphere map, then there exists a unitary U such that $Uf = \tilde{f} \oplus 0$, where \tilde{f} has linearly independent components. b) Find an example of a map of hyperquadrics whose components are not linearly independent where $\langle Vf(z), f(z) \rangle = 1$ for all z (hence for hyperquadric maps, matters are not as simple).*

We take $T \in U(n, 1)$ and consider

$$\langle ATz, Tz \rangle \langle JTz, Tz \rangle = \langle T^*ATz, z \rangle \langle Jz, z \rangle. \quad (3.48)$$

Therefore, if we can put a pair of Hermitian matrices into canonical form by $*$ -congruence by nonsingular matrices, where one of the canonical forms is J , then we have classified all maps from the sphere to any hyperquadric up to the right equivalence. If we then pick out only those canonical pairs that give maps to spheres, we have classified all degree 2 maps of spheres.

Classification of a pair of matrices has a long history (going back to Kronecker). The results we are interested in date to the 1930s. A survey by Lancaster and Rodman [LR05] is useful. For convenience let us give the result as formulated in a paper by Horn and Sergeichuk [HS06], whose canonical form is easy to work with. They also give an algorithm for computing the canonical form, so we also have an algorithm for producing a normal form for maps of hyperquadrics.

Before we can give the result of Horn and Sergeichuk, we first define the building block matrices used for their canonical form. Let us define M_n as the $n \times n$ matrix with ones on the subdiagonal and the superdiagonal, and similarly N_n as the $n \times n$ matrix with negative ones on the subdiagonal and ones on the super diagonal.

$$M_n \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}, \quad N_n \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 & & 0 \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & 0 \end{bmatrix}. \quad (3.49)$$

Let $J_n(\lambda)$ be the $n \times n$ Jordan block with eigenvalue λ (λ on the diagonal, ones on the superdiagonal). Finally define

$$\Delta_n(\alpha, \beta) \stackrel{\text{def}}{=} \begin{bmatrix} 0 & & & \alpha \\ & & \alpha & \beta \\ & \ddots & \ddots & \\ \alpha & \alpha & \beta & \\ \alpha & \beta & & 0 \end{bmatrix}. \quad (3.50)$$

We let I_n denote the $n \times n$ identity matrix.

Theorem 3.6.2 (Horn-Sergeichuk [HS06]). *Let A, B be a pair of Hermitian matrices. They are simultaneously $*$ -congruent to a direct sum of blocks of the following four types, determined uniquely up to permutation.*

- (i) (M_n, iN_n) ,
- (ii) $\pm(\Delta_n(1, 0), \Delta_n(\alpha, 1))$, $\alpha \in \mathbb{R}$,
- (iii) $\pm(\Delta_n(0, 1), \Delta_n(1, 0))$,
- (iv) $\left(\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(\alpha + i\beta)^* \\ J_n(\alpha + i\beta) & 0 \end{bmatrix} \right)$, $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta i \neq i$, $\beta > 0$.

When we write $\pm(\Delta_n(1, 0), \Delta_n(\alpha, 1))$ we mean either the pair $(\Delta_n(1, 0), \Delta_n(\alpha, 1))$ or the pair $(\Delta_n(-1, 0), \Delta_n(-\alpha, -1))$. Similarly for $\pm(\Delta_n(0, 1), \Delta_n(1, 0))$.

We take the pair (J, A) , where J defines the $HQ(n, 1)$. We will put the pair (J, A) into canonical form. If we can diagonalize J and A simultaneously by $*$ -congruence, then the map is equivalent to a monomial map allowing negative exponents. It is easy to see that if the map takes sphere to sphere, then no negative exponents are allowed.

Exercise 3.6.2: *A monomial CR map taking sphere to sphere has no negative exponents.*

So if (J, A) is simultaneously diagonalizable, we are done. Therefore, let us see what happens in the case when we cannot simultaneously diagonalize J and A . From now on, assume that at least one block in the canonical form is larger than 1×1 .

As J defines the sphere, all blocks except one in the canonical form for J must be 1×1 blocks. All the possible blocks of larger size have at least 1 negative eigenvalue, and hence there can only be one of these. Furthermore, J has no zero eigenvalues. Such blocks that have at most 1 negative eigenvalue and no zero eigenvalues are $M_2, \pm\Delta_2(1, 0), \Delta_3(1, 0), \begin{bmatrix} 0 & I_1 \\ I_1 & 0 \end{bmatrix}$. All these blocks are 2×2 or 3×3 . Therefore, it will be sufficient to simply consider source dimension $n = 2$. Only in two dimensions can the map possibly be non-monomial. Thus we let $n = 2$ in all that follows. Therefore, J and A are both 3×3 , and we compute all the normal forms for the Hermitian forms that arise from degree-two maps.

Note that $\begin{bmatrix} 0 & I_1 \\ I_1 & 0 \end{bmatrix} = M_2$, and iN_2 can be obtained by swapping variables and letting $\alpha + i\beta = i$ in the block $\begin{bmatrix} 0 & \alpha - i\beta \\ \alpha + i\beta & 0 \end{bmatrix}$. So we need not consider the block M_2 , but just need to allow $\alpha + i\beta = i$.

Let us compute. First take the situation corresponding to the block Δ_2 in J . That is we have the following canonical form for (J, A)

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & 1 \end{bmatrix} \right). \quad (3.51)$$

For $\alpha, \beta \in \mathbb{R}$. Take $\mathcal{Z} = (z_1^2, z_1z_2, z_1z_3, z_2^2, z_2z_3, z_3^2)$. The form corresponding to the product $\langle Az, z \rangle \langle Jz, z \rangle = \langle B\mathcal{Z}, \mathcal{Z} \rangle$ is

$$B = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha + \beta & 0 & 0 & 0 \\ 0 & \alpha + \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 & 2\beta & 1 \\ 0 & 0 & 0 & \beta & 1 & 0 \end{bmatrix}. \quad (3.52)$$

If $\alpha + \beta \neq 0$, then this form clearly has at least two negative eigenvalues (notice the block structure). If $\beta = -\alpha \neq 0$, then it is also not hard to see that there must still be at least two negative eigenvalues. When $\beta = \alpha = 0$, A has rank one, and we have a first degree map.

The forms for the related canonical pair $\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & -1 \end{bmatrix} \right)$ are similar.

Next pair to consider is

$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 1 & 0 \end{bmatrix} \right), \quad (3.53)$$

for $\alpha \in \mathbb{R}$. After multiplying the forms we get

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 2\alpha & 1 \\ 0 & 0 & 2\alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 2\alpha & 1 & 1 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.54)$$

This form also always has at least two negative eigenvalues. To see this fact, notice that the form has a zero eigenvalue only when $\alpha = 0$. Hence, it is enough to check the signature of the matrix for $\alpha = 0$.

Finally we get to the canonical pair

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta - i\gamma \\ 0 & \beta + i\gamma & 0 \end{bmatrix} \right), \quad (3.55)$$

for $\alpha, \beta \in \mathbb{R}$ and $\gamma > 0$. We note that we can just rescale to have $\gamma = 1$ if needed. After multiplying the forms we get

$$B = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha + \beta - i\gamma & 0 & 0 & 0 \\ 0 & \alpha + \beta + i\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta - i\gamma \\ 0 & 0 & 0 & 0 & 2\beta & 0 \\ 0 & 0 & 0 & \beta + i\gamma & 0 & 0 \end{bmatrix}. \quad (3.56)$$

Again, because of the block structure and the fact that $\gamma \neq 0$, we easily see that this form must have at least two negative eigenvalues. \square

Hence we have proved that all degree 2 sphere maps are monomial. We have already classified all monomial maps. There is one little piece missing. We need to show that two monomial maps taking zero to zero with linearly independent components (distinct monomials) are spherically equivalent if and only if they are equivalent up to swapping of variables, reordering coefficients and a diagonal unitary matrix. Let us prove this for degree two maps using the machinery we have developed.

One direction is simple of course, so let us concentrate on the hard direction. Suppose that f and g are monomial degree-two maps that are spherically equivalent. Write down the forms corresponding to $\|f(z)\|^2 - 1$ and $\|g(z)\|^2 - 1$ in homogeneous coordinates and find the matrix A_f for f and A_g for g . We notice that these matrices must be diagonal. They are in canonical form, and the canonical form is canonical up to permutation of the blocks (hence permutation of variables).

Hence after a permutation of variables we get that the diagonal entries of A_f and A_g are equal. The diagonal entries in A_f and A_g are the absolute values squared of the coefficients. We are done.

Let us state the result in inhomogeneous coordinates.

Theorem 3.6.3. *Let $f: S^{2n-1} \rightarrow S^{2N-1}$, $n \geq 2$, be a nonconstant rational CR map of degree 2 or less. Then f is spherically equivalent to a monomial map.*

In particular, f is equivalent to a map taking (z_1, \dots, z_n) to

$$\begin{aligned} &(\sqrt{t_1} z_1, \sqrt{t_2} z_2, \dots, \sqrt{t_n} z_n, \sqrt{1-t_1} z_1^2, \sqrt{1-t_2} z_2^2, \dots, \sqrt{1-t_n} z_n^2, \\ &\quad \sqrt{2-t_1-t_2} z_1 z_2, \sqrt{2-t_1-t_3} z_1 z_3, \dots, \sqrt{2-t_{n-1}-t_n} z_{n-1} z_n), \\ &0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1. \end{aligned} \quad (3.57)$$

Furthermore, all maps of the form (3.57) are mutually spherically inequivalent for different parameters (t_1, \dots, t_n) .

We note that for the special case $n = 2$ a classification theorem was proved by Ji and Zhang. The fact that the maps are all equivalent to monomial maps is new however. Furthermore, the moduli space is much easier to see in our formulation.

We also note one minor point from the proof. As long as the source is a sphere, we can let the target be any hyperquadric and we still obtain that such maps are equivalent to monomial maps except for perhaps two variables.

3.7 Monomial degree estimates

We end the course with a quick tour of the degree estimates proof for monomial maps. We prove only the two dimensional case here as the three dimensional case is much more complicated. The original proof of a degree estimate in $n = 2$ is due to D'Angelo-Kos-Riehl [DKR03]. The modified more general theorem proved below was done by Lebl-Peters [LP11]. We have noted that studying monomial CR sphere maps is equivalent to studying real polynomials $p(x, y)$ such that $p(x, y) = 1$ on $x + y = 1$ and such that p has positive coefficients. We will make one more change. We will work in the projective space, so we will add a homogenizing variable t . We will further replace t with $-t$ and study polynomials $P(x, y, t)$ such that $P(x, y, t) = 0$ on $x + y + t = 0$.

Definition 3.7.1. Denote by $\mathcal{S} = \mathcal{S}(2)$ the set of homogeneous polynomials $P(x, y, t)$ such that $P(x, y, t) = 0$ on $x + y + t = 0$. We say P is *irreducible* if P cannot be written as $P = P_1 + P_2$, where $P_1, P_2 \in \mathcal{S}$ and P_1 and P_2 share no common monomials. We define *p -degree* of P as the smallest integer d such that $P(x, y, t) = x^i y^j t^k R(x, y, t)$ for polynomial R of degree d . Let

$$N(P) = \# \text{ of distinct monomials in } P. \quad (3.58)$$

We will prove the following theorem.

Theorem 3.7.2. *If $P \in \mathcal{I}(2)$ is irreducible and of p -degree d , then*

$$d \leq 2N(P) - 5. \quad (3.59)$$

As a corollary we obtain.

Corollary 3.7.3. *Let $F: \mathbb{B}_2 \rightarrow \mathbb{B}_N$ be a monomial proper map of degree d . Then*

$$d \leq 2N - 3. \quad (3.60)$$

To obtain the corollary it suffices to prove the following proposition.

Proposition 3.7.4. *Let $p(x, y)$ be a polynomial of degree d with nonnegative coefficients such that $p(x, y) = 1$ when $x + y = 1$. When we obtain the induced $P \in \mathcal{I}$, P is irreducible and of p -degree d .*

Proof. If P were reducible, then the monomial corresponding to the constant term could only belong to one of the summands. Therefore, there would exist a polynomial $r(x, y)$ with nonnegative coefficients such that $r(x, y) = 0$ when $x + y = 1$. It is not hard to see that $r \equiv 0$. Thus P is irreducible. When we homogenize, there will exist a t^d and also a monomial of degree d that does not depend on t . Hence p -degree is d . \square

Definition 3.7.5. Let Q be the homogeneous polynomial

$$Q(x, y, t) = \frac{P(x, y, t)}{x + y + t}. \quad (3.61)$$

We define the function $D: \mathbb{Z}^2 \rightarrow \{0, P, N\}$ as follows. Let $c_{j,k}$ be the coefficient of $x^j y^k t^{d-j-k}$ in Q . Then

$$D(j, k) \stackrel{\text{def}}{=} \begin{cases} P & \text{if } c_{j,k} > 0, \\ 0 & \text{if } c_{j,k} = 0, \\ N & \text{if } c_{j,k} < 0. \end{cases} \quad (3.62)$$

We call D the *Newton diagram* of Q , and we will say that D is the Newton diagram corresponding to P .

We will call the $m \in \mathbb{Z}^2$ points of D , and we will call m a 0-point if $D(m) = 0$, a P -point if $D(m) = P$ and an N -point if $D(m) = N$.

Therefore, the Newton diagram of Q is an array of P s, N s, and 0 s one for each coefficient of Q . The condition of irreducibility will turn out to be a connectedness condition on the Newton diagram of Q and the p -degree will be related to the size of the set of nonzero points.

We will generally ignore points $m \in \mathbb{Z}^2$ where $D(m) = 0$. We can give graphical representation of D by drawing a lattice, and then drawing the values of D in the lattice. For convenience, when drawing the lattice, we put $(0, 0)$ at the origin, and then let $(0, 1)$ be directed at angle $\frac{\pi}{3}$ and $(1, 0)$ at angle $\frac{2\pi}{3}$. In the figures we will use light color circles for positive coefficients and dark color circles

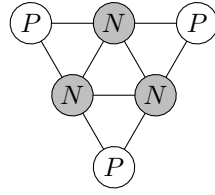


Figure 3.2: Newton diagram for $Q(x, y, t) = x^2 + y^2 + t^2 - xy - yt - xt$.

for negative coefficients. We will not draw the circles corresponding to the zero coefficients. See Figure 3.2 for an example.

We will say that two points (or monomials) are *adjacent* if the corresponding monomials m_1 and m_2 satisfy $v_1 m_1 = v_2 m_2$, where v_1 is $x, y,$ or t and v_2 is $x, y,$ or t . Between any two points (or any two monomials) we can define the distance $\text{dist}(m_1, m_2)$ as 1 if the corresponding monomials are adjacent, and if they are not adjacent as the length of the shortest path along adjacent monomials.

The *support* $K \subset \mathbb{Z}^2$ of a Newton diagram D is the set of nonzero points. That is, $K = D^{-1}(\{P, N\})$. We say that the support K of a Newton diagram D is *connected* if any two points of K are joined by a path of adjacent points of K . Let $\#(D)$ be the number of nonzero points in D , in other words, the cardinality of K .

For a set $K \subset \mathbb{Z}^2$, we let $\widehat{K} \subset \mathbb{Z}^2$ be the smallest simplex that contains K . More precisely, $\widehat{K} \subset \mathbb{Z}^2$ is the smallest set such that $K \subset \widehat{K}$ and for some $(j, k) \in \mathbb{Z}^2$ and some $m \in \mathbb{N}$ we have

$$\widehat{K} = \{(a, b) \in \mathbb{Z}^2 : a \geq j, b \geq k, a + b \leq m\}. \quad (3.63)$$

When $K \subset \mathbb{N}_0^2$ then the diameter of \widehat{K} in the distance defined above is easily seen to be $m - j - k$. We define the *size* of K as the diameter of \widehat{K} plus 1. Therefore the size of K as above is $m - j - k + 1$.

Proposition 3.7.6. $P \in \mathcal{S}$ is irreducible then D is connected.

Proof. If D is not connected, take corresponding monomials for the two components multiply by $(x + y + t)$ and note that we get two polynomials in \mathcal{S} with no monomials in common. \square

Definition 3.7.7. Let $D: \mathbb{Z}^2 \rightarrow \{0, P, N\}$ be a Newton diagram. Take $(j, k) \in \mathbb{Z}^2$ and let

$$E = E(j, k) \stackrel{\text{def}}{=} \{(j, k), (j - 1, k), (j, k - 1)\}. \quad (3.64)$$

We call E a *node* if $D(E) \subset \{0, P\}$ or $D(E) \subset \{0, N\}$, but $D(E) \neq \{0\}$. For a diagram D let

$$\#(D) \stackrel{\text{def}}{=} (\text{number of nodes in } D). \quad (3.65)$$

See Figure 3.3 for an example of a diagram with nodes marked. In dimension 2, the nodes are marked with a triangle where each vertex of the triangle touches a point of the node. In dimension 3 we mark the nodes similarly with a simplex.

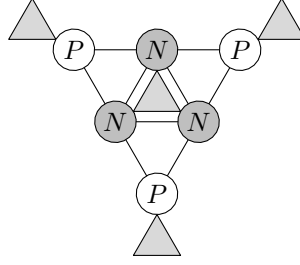


Figure 3.3: Nodes in Newton diagrams.

The key point is that if the diagram of polynomial $P \in \mathcal{S}$ has a node $E(j, k)$, then there must exist a term in P as the only terms of Q that can contribute to this term are those corresponding to $E(j, k)$.

Proposition 3.7.8. *Let D be a Newton diagram with support K in two variables. Suppose that K is of size d , is connected, contains $(0, 0)$. Then*

$$\#(D) \geq \frac{d+5}{2}. \quad (3.66)$$

The proof of Proposition 3.7.8 will consist of two short lemmas. We will say that a diagram D' is an *extension* of D if the support K' of D' contains the support K of D and such that $D'|_K = D|_K$.

Lemma 3.7.9. *Let D and K be as in Proposition 3.7.8. Let $m < d$ and suppose that K contains all points (j, k) with $j+k \leq m-1$. Then there exists a Newton diagram D' with support $K' \supset K$ such that $D'|_K = D$ and such that $\#(D') \leq \#(D)$ and such that K' contains all points (j, k) with $j+k \leq m$.*

Proof. We change the 0-points at level k into alternating N and P -points, one connected group of 0-points of level k at a time. There must be at least one P or N -point at the k -level. There are two cases that we should consider. In the first case, the group of 0-points has N or P -points on both sides. In this case no new nodes can possibly be created.

If the group goes all the way to the end of the positive quadrant, then we have to consider the two possible alternating strings of P s and N s. One of them will not create a new node at the point at the boundary of the positive quadrant. See Figure 3.4. The circles with the dark borders are the nonzero points that were added in a single step. \square

By induction using Lemma 3.7.9 we obtain a diagram with support $K = \{(a, b) \in \mathbb{N}_0^2 \mid a+b < d\}$. Proposition 3.7.8 now follows from the following lemma.

Lemma 3.7.10. *Let D be a Newton diagram in two variables with support K of size d and suppose that K contains all $(a, b) \in \mathbb{N}_0^2$ with $a+b < d$. Then D has at least $\frac{d+5}{2}$ nodes.*

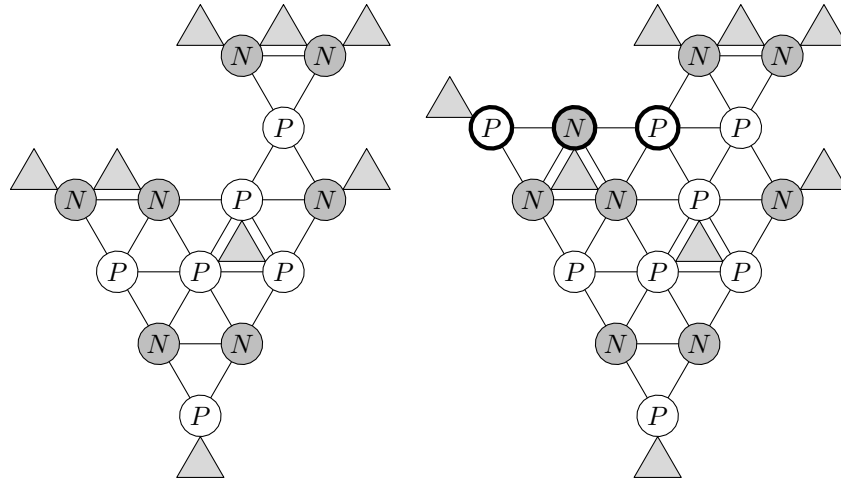


Figure 3.4: Filling a 2-dimensional diagram.

Proof. First count the number of nodes that involve points at level d (all 0-points) or $d - 1$ (no 0-points). The number of nodes is $d + 1 - k$ where k is the number of sign changes at the $d - 1$ level.

Now look at two adjacent levels less than d . If the number of sign changes in one level is m and the number of sign changes in the lower level is n , then there must be at least $\max\{\frac{m-n}{2}, 0\}$ nodes involving points only on these two levels. Now by induction plus adding the node at the bottom will complete the proof. \square

Finally we prove the following lemma that at once implies Theorem 3.7.2 as soon as we note that the number of nodes bounds the number of terms in P .

Lemma 3.7.11. *Let D be a Newton diagram with connected support K of size d in two variables. Then*

$$\#(D) \geq \frac{d+5}{2}. \tag{3.67}$$

Proof. Without loss of generality we may assume that the support K contains a point $(a, 0)$ and a point $(0, b)$ for some a and b . By connectedness K must contain a connected path from $(a, 0)$ to $(0, b)$. By the proof of Lemma 3.7.9 we can change all the 0-points of D above this connected path into P and N -points without increasing the number of nodes. By rotating the diagram swapping the roles of the variables and repeating the procedure we obtain a filled in diagram so that we can apply Proposition 3.7.8 to complete the proof. \square

Further Reading

Here are useful books for extra reading, some were cited. Papers cited appear in the next section.

- [AG93] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Dover Publications Inc., New York, 1993. MR1255973
- [BER99] M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999. MR1668103
- [Bog91] Albert Boggess, *CR manifolds and the tangential Cauchy-Riemann complex*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1991. MR1211412
- [Chi89] E. M. Chirka, *Complex analytic sets*, Mathematics and its Applications (Soviet Series), vol. 46, Kluwer Academic Publishers Group, Dordrecht, 1989. MR1111477
- [D'A93] John P. D'Angelo, *Several complex variables and the geometry of real hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993. MR1224231
- [Hör90] Lars Hörmander, *An introduction to complex analysis in several variables*, 3rd ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990. MR1045639
- [HJ85] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985. MR832183
- [Kra92] Steven G. Krantz, *Function theory of several complex variables*, 2nd ed., The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992. MR1162310
- [Rud91] Walter Rudin, *Functional analysis*, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991. MR1157815
- [Rud80] _____, *Function theory in the unit ball of \mathbb{C}^n* , Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York, 1980. MR601594
- [Whi72] Hassler Whitney, *Complex analytic varieties*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1972. MR0387634

Bibliography

- [CS83] J. A. Cima and T. J. Suffridge, *A reflection principle with applications to proper holomorphic mappings*, Math. Ann. **265** (1983), no. 4, 489–500, DOI 10.1007/BF01455949. MR721883
- [DKR03] John P. D’Angelo, Šimon Kos, and Emily Riehl, *A sharp bound for the degree of proper monomial mappings between balls*, J. Geom. Anal. **13** (2003), no. 4, 581–593, DOI 10.1007/BF02921879. MR2005154
- [DF78] Klas Diederich and John E. Fornaess, *Pseudoconvex domains with real-analytic boundary*, Ann. Math. (2) **107** (1978), no. 2, 371–384. MR0477153
- [Dor90] Avner Dor, *Proper holomorphic maps between balls in one co-dimension*, Ark. Mat. **28** (1990), no. 1, 49–100, DOI 10.1007/BF02387366. MR1049642
- [Far82] James J. Faran, *Maps from the two-ball to the three-ball*, Invent. Math. **68** (1982), no. 3, 441–475, DOI 10.1007/BF01389412. MR669425
- [Far86] ———, *The linearity of proper holomorphic maps between balls in the low codimension case*, J. Differential Geom. **24** (1986), no. 1, 15–17. MR857373
- [For89] Franc Forstnerič, *Extending proper holomorphic mappings of positive codimension*, Invent. Math. **95** (1989), no. 1, 31–61, DOI 10.1007/BF01394144. MR969413
- [Ham05] Hidetaka Hamada, *Rational proper holomorphic maps from \mathbf{B}^n into \mathbf{B}^{2n}* , Math. Ann. **331** (2005), no. 3, 693–711, DOI 10.1007/s00208-004-0606-2. MR2122546
- [HS06] Roger A. Horn and Vladimir V. Sergeichuk, *Canonical forms for complex matrix congruence and *congruence*, Linear Algebra Appl. **416** (2006), no. 2-3, 1010–1032, DOI 10.1016/j.laa.2006.01.005. MR2242477
- [Hua99] Xiaojun Huang, *On a linearity problem for proper holomorphic maps between balls in complex spaces of different dimensions*, J. Differential Geom. **51** (1999), no. 1, 13–33. MR1703603
- [HJ01] Xiaojun Huang and Shanyu Ji, *Mapping \mathbf{B}^n into \mathbf{B}^{2n-1}* , Invent. Math. **145** (2001), no. 2, 219–250, DOI 10.1007/s002220100140. MR1872546

- [HJX06] Xiaojun Huang, Shanyu Ji, and Dekang Xu, *A new gap phenomenon for proper holomorphic mappings from B^n into B^N* , *Math. Res. Lett.* **13** (2006), no. 4, 515–529. MR2250487
- [LR05] Peter Lancaster and Leiba Rodman, *Canonical forms for Hermitian matrix pairs under strict equivalence and congruence*, *SIAM Rev.* **47** (2005), no. 3, 407–443, DOI 10.1137/S003614450444556X. MR2178635
- [LP11] Jiří Lebl and Han Peters, *Polynomials constant on a hyperplane and CR maps of hyperquadrics*, *Mosc. Math. J.* **11** (2011), no. 2, 287–317. arXiv:0910.2673. MR2859238
- [Web79] S. M. Webster, *On mapping an n -ball into an $(n + 1)$ -ball in complex spaces*, *Pacific J. Math.* **81** (1979), no. 1, 267–272. MR543749

Index

- analytic curve, 18
- antiholomorphic, 9
- antiholomorphic coordinates, 5

- bidegree, 44
- bihomogeneous, 44

- Cauchy estimates, 7
- Cauchy integral formula in several variables, 6
- center of a polydisc, 5
- codimension, 16
- complete intersection, 16
- complex projective space, 44
- complex subvariety, 14
- complex variety, 14
- complexification, 23

- dimension, 16
- dimension at a regular point, 15
- discriminant function, 13
- discriminant set, 13

- Euclidean inner product, 5
- Euclidean norm, 5

- generalized special unitary group, 48
- generalized unitary group, 48
- generalized Whitney maps, 41
- germ of a function, 8
- germ of a set, 9

- Hermitian symmetric polynomial, 23
- holomorphic, 6
- holomorphic coordinates, 5
- homogeneous, 44

- homogeneous coordinates, 44
- hypervarieties, 16

- Identity theorem, 7
- inhomogeneous coordinates, 44
- irreducible, 17
- irreducible components, 17
- irreducible polynomial in \mathcal{I} , 53

- Levi-flat, 34
- Lewy hypersurface, 27
- Local parametrization theorem, 17
- local variety, 14

- matrix of coefficients, 23
- Maximum principle, 8
- monomial mapping, 39
- multi-index notation, 6

- order of vanishing, 8

- p-degree, 53
- polarization, 22
- polydisc, 5
- polyradius of a polydisc, 5
- projective space, 44
- proper mapping, 20

- radius of a polydisc, 5
- real-algebraic hypersurface, 26
- real-analytic hypersurface, 33
- reducible, 17
- regular point, 15

- Segre variety, 27, 34

signature pair, 24

simple point, 15

singular point, 15

spherically equivalent, 39

subvariety, 14

variety, 14

Weierstrass division theorem, 11

Weierstrass polynomial, 9

Weierstrass preparation theorem, 10

Whitney map, 41