

BA: 0.3 part 3

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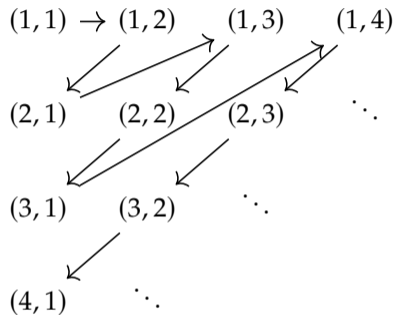
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If A is not countable, then it is *uncountable*.

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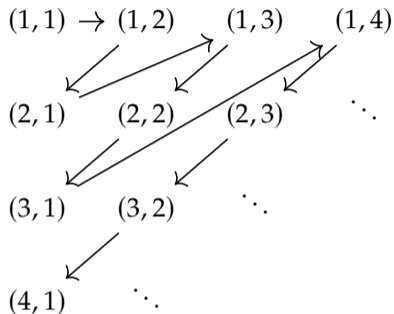
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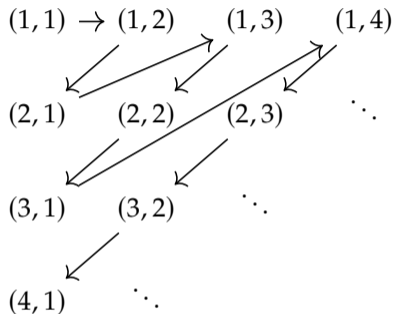
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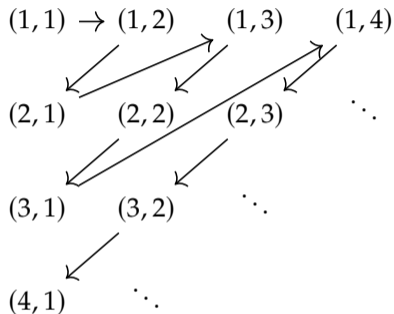


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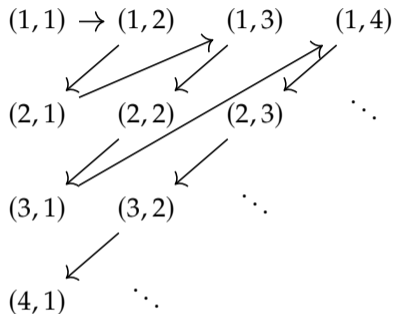
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For all \mathbb{Q} , also include 0 and the negatives: $0, 1/1, -1/1, 1/2, -1/2, 2/1, -2/1$, etc.

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Interestingly, it works for infinite sets too:

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In fact, $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$, etc.